Meromorphic Potentials and Smooth CMC Surfaces

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1 Introduction

In the recent paper [5] it was shown that every conformal CMC immersion $\Phi : D \to \mathbb{R}^3$, $D \subset \mathbb{C}$ the whole complex plane or the open unit disk, can be produced from a meromorphic matrix valued one form

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & f(z) \\ g(z) & 0 \end{pmatrix} \mathrm{d}z, \tag{1.1.1}$$

$\lambda \in S^1$, the so called "meromorphic potential".

Here f and g are meromorphic functions of $z \in D$ and $f(z)g(z) = E(z)$, where $E(z)\mathrm{d}z^2$ is up to a constant factor (see the appendix) the Hopf differential of the surface $M = (D, \Phi)$.

Moreover, the normalizations used in [5] imply, that f and g don't have any poles at $z = 0$.

The construction involves the following steps:

1. Solve the initial value problem

$$\mathrm{d}g_- = g_-\xi,\ g_-(z,\lambda) \in \Lambda_*^-\mathbf{SL}(2,\mathbb{C})_\sigma,\ g_-(0,\lambda) = I. \tag{1.1.2}$$

2. Compute an Iwasawa decomposition of g_-:

$$g_- = Fg_+^{-1},\ F \in \Lambda\mathbf{SU}(2)_\sigma,\ g_+ \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma, \tag{1.1.3}$$

   where $g_+(\lambda = 0)$ is of the form $\mathrm{diag}(a, a^{-1})$ with a an arbitrary complex number as opposed to the traditional Iwasawa decomposition, where $|a| = 1$.

3. Apply the Sym-Bobenko formula,

$$\phi_\lambda(z) = \frac{\mathrm{d}}{\mathrm{d}\theta} F \cdot F^{-1} + \frac{1}{2} F \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} F^{-1},\ \lambda = e^{i\theta}, \tag{1.1.4}$$

   to $F = F(e^{i\theta}, z)$ to obtain CMC immersions with mean curvature $H = -\frac{1}{2}$. This actually yields a CMC immersion for every $\lambda \in S^1$ (see A.7). The immersion $\Phi = \Phi_1$ is the given one.

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Here $\Lambda\mathbf{SL}(2,\mathbb{C})_\sigma$ and $\Lambda\mathbf{SU}(2)_\sigma$ denote the twisted loop groups over $\mathbf{SL}(2,\mathbb{C})$ and $\mathbf{SU}(2)$, respectively, given by the automorphism

$$\sigma : g \mapsto (\mathrm{Ad}\sigma_3)(g),\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

e.g.

$$\Lambda\mathbf{SL}(2,\mathbb{C})_\sigma = \{g : S^1 \longrightarrow \mathbf{SL}(2,\mathbb{C}) |\ g(-\lambda) = \sigma(g(\lambda))\}.$$

In order to make these loop groups complex Banach Lie groups, we equip them, as in [5], with some H^s-topology for $s > \frac{1}{2}$.

Elements of these twisted loop groups are matrices with offdiagonal entries which are odd functions and diagonal entries which are even functions in the parameter λ. All entries are in the Banach algebra $\mathcal{A}$ of H^s-smooth functions.

Their Lie-algebras are then complex Banach Lie algebras, e.g.

$$\Lambda\mathbf{sl}(2,\mathbb{C})_\sigma = \{x : S^1 \longrightarrow \mathbf{sl}(2,\mathbb{C}) |\ x(-\lambda) = \sigma(x(\lambda)),\ x \text{ is } H^s\text{-smooth}\}.$$

The indices $+$ and $-$ refer to the usual splitting of the loop group into maps analytic inside and outside the unit circle. In addition by $\Lambda^-_*\mathbf{SL}(2,\mathbb{C})$ we denote the subgroup of elements of $\Lambda^-\mathbf{SL}(2,\mathbb{C})$, that take the value I at infinity. Then the set $\Lambda^-_*\mathbf{SL}(2,\mathbb{C})_\sigma \cdot \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$ is open and dense in $\Lambda\mathbf{SL}(2,\mathbb{C})_\sigma$.

The decomposition,

$$\Lambda\mathbf{SL}(2,\mathbb{C})_\sigma \cong \Lambda\mathbf{SU}(2)_\sigma \times \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma,$$

is defined for all elements of $\Lambda\mathbf{SL}(2,\mathbb{C})_\sigma$.

To be more precise, in [5] the following twisted versions of well known theorems [7] are proved:

(i) For each solvable subgroup B of $\mathbf{SL}(2,\mathbb{C})$ with $\mathbf{SU}(2) \cap B = \{I\}$, multiplication

$$\Lambda\mathbf{SU}(2)_\sigma \times \Lambda^+_B\mathbf{SL}(2,\mathbb{C})_\sigma \longrightarrow \Lambda\mathbf{SL}(2,\mathbb{C})_\sigma$$

is a diffeomorphism onto. Here $\Lambda^+_B\mathbf{SL}(2,\mathbb{C})_\sigma$ denotes the set of all elements $g_+(\lambda)$ of $\Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$ with $g_+(0) \in B$.

(ii) Multiplication

$$\Lambda^-_*\mathbf{SL}(2,\mathbb{C})_\sigma \times \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma \longrightarrow \Lambda\mathbf{SL}(2,\mathbb{C})_\sigma$$

is a diffeomorphism onto the open and dense subset $\Lambda^-_*\mathbf{SL}(2,\mathbb{C})_\sigma \cdot \Lambda\mathbf{SL}^+(2,\mathbb{C})_\sigma$ of $\Lambda\mathbf{SL}(2,\mathbb{C})_\sigma$, called the "big cell" [8].

If at a point $z_0 \in D$, f has a zero and g is holomorphic, then, as was pointed out in [5] the map Φ_1 fails to be an immersion at $z = z_0$ (see also the appendix).

If $g(z)$ has a zero at $z = z_0$, and f is defined and nonzero there, then one gets an umbilic at the image of z_0 in $\mathbb{R}^3$.

This allows one to use the DPW method, to construct a CMC immersion with a prescribed distribution of umbilics in the domain D.

Special cases involve the sphere (minus a point), where $E(z) = 0$, $f = 1$, the cylinder with $E(z) = f(z) = 1$ and the generalized Smyth surfaces, where $E(z)$ is a polynomial whose roots give the umbilics, and $f(z) = 1$.

These cases are among many investigated and visualized by several groups using the Bubbleman AVS network by Ulrich Pinkall and Charlie Gunn, which is based on earlier work of D. Lerner and I. Sterling [6]. In all cases the functions $f(z)$ and $g(z)$ were chosen holomorphic in the entire domain D.

If we start with a meromorphic function $f(z)$ and a nonvanishing holomorphic Hopf differential $E(z)\mathrm{d}z^2$, each of the three steps in the construction imposes conditions on the choice of the meromorphic potential, i.e. on $f(z)$ and $E(z)$. In order to get a smooth surface, we first need to find a meromorphic solution $g_-(z,\lambda) \in \Lambda_*^- \mathbf{SL}(2,\mathbb{C})_\sigma$ to (1.1.2), which will turn out not to be possible for arbitrary functions $f(z)$ and $E(z)$.

In the second step the factor F in the Iwasawa decomposition needs to be smooth, which imposes further conditions on $f(z)$ and $E(z)$.

Finally, applying the Sym-Bobenko formula (see the appendix), we obtain a map $\Phi = \Phi_1$, but one can get branch points of the surface defined by Φ, i.e. $\Phi = \Phi_1$ possibly fails to be an immersion at certain points of D.

In this note we will give necessary and sufficient conditions on $f(z)$ and $E(z)$ for Φ, the CMC map associated with the meromorphic potential ξ, to be a CMC immersion. Then the surface $M = (D, \Phi)$ will be smooth without branchpoints. On the other hand, if M is a smooth immersed CMC surface without branchpoints, then there always exists a CMC immersion Φ and a subset D of $\mathbb{C}$, such that $M = (D, \Phi)$. Therefore, we locally describe all smooth immersed CMC surfaces in $\mathbb{R}^3$ without branchpoints.

In section 2 we will develop a necessary and sufficient algebraic condition for f and the Hopf differential, which ensures that there exists a meromorphic matrix solution of (1.1.2) in the proper twisted loop group.

While this gives us a meromorphic mapping $g_-(z,\lambda)$ from D into the loop group $\Lambda_*^- \mathbf{SL}(2,\mathbb{C})_\sigma$, it does not guarantee the smoothness of the compact part $F(z,\lambda)$ of the Iwasawa decomposition (1.1.3) of $g_-(z,\lambda)$.

Necessary and sufficient local conditions for this are derived in section 3. These take the shape of compatibility conditions on the pole and zero orders of the functions $f(z)$ and $E(z)$. Incorporated are also conditions for the (non)existence of branch points.

In all our investigations we will exclude the trivial case $E = 0$, the case of the round sphere.

In section 4 we will give three examples of nonholomorphic meromorphic potentials, which are associated with CMC immersions.

We close with an appendix, in which we fix the conventions used in this paper.

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The pictures at the end of the paper were produced using a modification of the Bubbleman AVS network written by Charlie Gunn at the SFB-288, TU-Berlin.

2 Meromorphic Solutions of $g_-^{-1}\mathrm{d}g_- = \xi$

2.1 In this section we want to find a meromorphic matrix solution $g_-(z,\lambda)$ of the initial value problem

$$g_-' = g_-\xi, \tag{2.1.1}$$
$$g_-(0,\lambda) = I. \tag{2.1.2}$$

Here $(\cdot)'$ denotes the derivative w.r.t. the complex variable z and

$$\xi = \lambda^{-1}Q(z) = \lambda^{-1}\begin{pmatrix} 0 & f \\ g = f^{-1}E & 0 \end{pmatrix}, \tag{2.1.3}$$

i.e. we have omitted the $\mathrm{d}z$ factor in equation (1.1.2).

We note that then automatically $g_-(z,\lambda) \in \Lambda_*^- \mathbf{SL}(2,\mathbb{C})_\sigma$, i.e. $\det g_-(z,\lambda) = 1$, the diagonal entries are even functions and the offdiagonal entries are odd functions of λ, and $g_-(z,\infty) = I$.

It is important to note that the equation (2.1.1) is invariant under coordinate changes. More precisely, if f and E are the defining data for Q in the coordinate z, then in a different coordinate w the corresponding functions $\hat{f}(w)$ and $\hat{E}(w)$ are given by

$$\begin{aligned} \hat{f}(w) &= f(z(w))\frac{\mathrm{d}z}{\mathrm{d}w}, & (2.1.4)\\ \hat{E}(w) &= E(z(w))\left(\frac{\mathrm{d}z}{\mathrm{d}w}\right)^2. & (2.1.5) \end{aligned}$$

If in addition $w = 0 \Leftrightarrow z = 0$, then also the equation (2.1.2) is valid in both coordinate systems.

This allows the following normalization of f: Locally around a point z_0 we can always choose the coordinate w to be such that $\hat{f}(w) = (w - w_0)^k$ or $\hat{f}(w) = (w - w_0)^{-k}$, where $w_0 = w(z_0)$ and $k \geq 0$ is the pole/zero-order of f at z_0.

Next we look at the matrix entries of g_-:

$$g_- = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.1.6}$$

Equation (2.1.1) can be written as a set of four scalar differential equations:

$$\begin{aligned} \lambda a' &= bf^{-1}E, & (2.1.7)\\ \lambda b' &= af, & (2.1.8)\\ \lambda c' &= df^{-1}E, & (2.1.9)\\ \lambda d' &= cf, & (2.1.10) \end{aligned}$$

and the initial condition (2.1.2) translates to

$$a(0,\lambda) = d(0,\lambda) \quad = \quad 1, \tag{2.1.11}$$
$$b(0,\lambda) = c(0,\lambda) \quad = \quad 0. \tag{2.1.12}$$

We differentiate equations (2.1.8) and (2.1.10) and substitute for a' and c' using equations (2.1.7) and (2.1.9), respectively. It follows that b and d solve

$$y'' - \frac{f'}{f}y' - \lambda^{-2}Ey = 0. \tag{2.1.13}$$

Analogously we show that a and c solve

$$y'' + \left(\frac{f'}{f} - \frac{E'}{E}\right)y' - \lambda^{-2}Ey = 0. \tag{2.1.14}$$

The initial conditions are given by (2.1.11) and (2.1.12).

2.2 Note that (2.1.14) can also be stated as

$$y'' - \frac{g'}{g}y' - \lambda^{-2}Ey = 0. \tag{2.2.1}$$

Remark: In this paper we will use methods and results of [5]. Therefore we need that all functions of λ are in the algebra $\mathcal{A} = H^s$, $s > \frac{1}{2}$. We note that the coefficients of the differential equations above are, as functions of λ, obviously contained in the algebra $\mathcal{A}$. Therefore by the standard theory of differential equations in Banach spaces (see e.g. [3]) the solutions $y = y(z,\lambda)$ are, as functions of λ, analytic and contained in $\mathcal{A}$ as well. Moreover, if we assume $y(0,\lambda) = 1$, then y has an expansion relative to λ of the form

$$y = \sum_{n=0}^{\infty} q_n \lambda^{-2n}. \tag{2.2.2}$$

Here the coefficients are functions of z. If y is meromorphic in z, then the $q_n = q_n(z)$ are meromorphic as well. So all the functions occurring in this paper will have values in the Banach algebra $\mathcal{A} = H^s$, $s > \frac{1}{2}$, as required by [5].

Theorem: *Let f and g be meromorphic functions without poles at $z = 0$. The initial value problem (2.1.1), (2.1.2) with meromorphic potential (1.1.1) has a global meromorphic matrix solution g_- if and only if (2.1.13) has at every point in D two linearly independent local meromorphic solutions.*

We will postpone the proof of this theorem until section 2.7.

Remark: In view of equations (2.1.8) and (2.1.10) we note that every solution of (2.1.13) induces the solution $f^{-1}y'$ of (2.1.14). Similarly every solution of (2.1.14) produces a solution of (2.1.13). Moreover the existence of two linearly independent meromorphic solutions of (2.1.13) is equivalent to the existence of two meromorphic solutions of (2.1.14).

2.3 Theorem 2.2 shows, that it suffices to investigate (2.1.13).

If f has neither poles nor zeroes around $z = z_0$, the second order differential equation (2.1.13) has locally holomorphic coefficients. Therefore there exist locally two linearly independent holomorphic solutions. So let us look at the poles and zeroes of f.

We will first prove a theorem which allows us to restrict our attention to poles and zeroes of even order.

Theorem: *The function f in the meromorphic potential is the square of a meromorphic function.*

Proof: By [4] there exists for a given CMC immersion Φ a holomorphic potential η of the form

$$\eta(z, \lambda) = \lambda^{-1} \begin{pmatrix} 0 & c \\ c^{-1}E(z) & 0 \end{pmatrix} + \eta_+(z, \lambda),\ c \in \mathbb{C},\ c \neq 0 \tag{2.3.1}$$

where $\eta_+(z, \lambda) \in \Lambda^+\mathbf{sl}(n, \mathbb{C})_\sigma$. The Birkhoff splitting $g = g_- g_+$ of the integral g of η produces the meromorphic potential. But this amounts to a gauge transformation of η with g_+^{-1}. As g_+ contains no negative powers of λ, the λ^{-1} coefficient of $g_-^{-1}\mathrm{d}g_-$ is the λ^{-1} coefficient of η conjugated with the constant term g_0 of g. If we write $g_0 = \mathrm{diag}(\omega_0, \omega_0^{-1})$, then the meromorphic potential is given by

$$\xi = g_-^{-1}\mathrm{d}g_- = g_+ \eta g_+^{-1} - \mathrm{d}g_+ g_+^{-1} = \lambda^{-1} \begin{pmatrix} 0 & cw_0^2 \\ c^{-1}w_0^{-2}E & 0 \end{pmatrix} \mathrm{d}z. \tag{2.3.2}$$

and $f = cw_0^2$. $\square$

It follows directly:

Corollary: *f has only poles and zeroes of even order.*

2.4 Assume f has a pole of order $n \geq 2$ at some point $z = z_0$. Then (2.1.13) has a regular singular point at $z = z_0$ and the form of the solutions can be decided by looking at the indicial equation

$$r(r-1) + nr = 0 \tag{2.4.1}$$

The roots of this equation are

$$r_1 = 0,\ r_2 = -n + 1. \tag{2.4.2}$$

The theory of ordinary differential equations with regular singular points (see e.g. [2]) shows that there always exists a locally meromorphic solution y_1 of (2.1.13). This solution is associated with the higher of the two roots. It is also well known, that every formal power series solution of (2.1.13) converges.

Making the ansatz $y_2 = y_1 \cdot v$ for a second linearly independent solution y_2, we get

$$v'' = \left(\ln \frac{f}{y_1^2}\right)' v', \tag{2.4.3}$$

or

$$v' = C \frac{f}{y_1^2},\ C = \mathrm{const} \neq 0 \tag{2.4.4}$$

In general this second solution will therefore have a logarithmic singularity at $z = z_0$, unless the residue of the right hand side of (2.4.4) vanishes.

$$\oint_{z_0} \frac{f}{y_1^2} \mathrm{d}z = \mathrm{res}_{z=z_0} \frac{f}{y_1^2} = 0. \tag{2.4.5}$$

The solution y_1 can be written as a power series ($w = z - z_0$)

$$y_1 = \sum_{i \geq 0} a_i w^i,\ a_0 = 1, \tag{2.4.6}$$

which has has neither a pole nor a zero at z_0. y_1 is a nontrivial holomorphic function near $z = z_0$.

2.5 Let us now assume that f has a zero of order $n \geq 2$ at $z = z_0$. This case is only interesting if f doesn't divide the Hopf differential E, because otherwise g is also nonsingular at $z = z_0$, and we obtain two holomorphic solutions of (2.1.13).

Then in analogy to the last section, one gets the indicial equation

$$r(r-1) - nr = 0, \tag{2.5.1}$$

with roots

$$r_1 = n + 1,\ r_2 = 0. \tag{2.5.2}$$

We denote by y_1 the solution associated with the larger root r_1. It is locally holomorphic and of the form ($w = z - z_0$)

$$y_1 = \sum_{i \geq 0} a_i w^{i+n+1},\ a_0 = 1. \tag{2.5.3}$$

The ansatz $y_2 = y_1 \cdot v$ again yields condition (2.4.5) for the second solution y_2 to be meromorphic. y_2 is then also locally holomorphic at $z = z_0$.

Theorem: *If f has a pole or zero at $z = z_0$ and if y_1 denotes a meromorphic solution to (2.1.13), then the following is equivalent:*

a) *(2.1.13) has locally two linearly independent meromorphic solutions.*

b) *The solution y_1 satisfies equation (2.4.5).*

Proof: By the calculations in the last two sections we know, that for every meromorphic solutions y_1, there is a second solution y_2, given by $y_2 = y_1 \cdot v$, where v satisfies (2.4.4). This solution is linearly independent of y_1, since f and therefore v and v' don't vanish identically. Moreover, y_2 is also meromorphic, if and only if (2.4.5) is satisfied. This proves the theorem. □

Remark: By equation (2.1.4) condition (2.4.5) is invariant under coordinate changes $z \to w(z)$.

2.6 The condition (2.4.5) for the existence of two linearly independent meromorphic solutions of (2.1.13) can be expressed more explicitly in terms of E and f.

In general y_2 will be of the form $(w = z - z_0)$

$$y_2(w) = \alpha y_1(w) \ln w + w^{r_2} \left(1 + \sum_{k=1}^{\infty} \tilde{a}_k w^k \right). \tag{2.6.1}$$

If (2.4.5) is satisfied, then the coefficient α of the logarithmic term vanishes and y_2 is of the form

$$y_2 = w^{r_2} + \sum_{k=1}^{\infty} \tilde{a}_k w^{k+r_2}. \tag{2.6.2}$$

We would also like to note that, if (2.1.13) has a solution of the form (2.6.2) even only formally, where r_2 is the smaller of the two roots of the indicial equation, then locally there exist two linearly independent meromorphic solutions.

Substituting (2.6.2) into (2.1.13) yields the recursion relation $(k \geq 2)$

$$\begin{aligned} \varphi(k+r_2)\tilde{a}_k &= -\sum_{s=0}^{k-1}(s+r_2)q_{k-s}\tilde{a}_s + \lambda^{-2}\sum_{s=2}^{k} E_{s-2}\tilde{a}_{k-s}, \\ \varphi(r_2+1)\tilde{a}_1 &= -r_2 q_1 \tilde{a}_0, \end{aligned} \tag{2.6.3}$$

for the coefficients $\tilde{a}_k$, where φ is the left hand side of the indicial equation, i.e.

$$\varphi(m) = m(m-1) + nm, \tag{2.6.4}$$

if $z = z_0$ is a pole of f of order n, and

$$\varphi(m) = m(m-1) - nm, \tag{2.6.5}$$

if $z = z_0$ is a zero of f of order n.

Moreover, the coefficients q_j are defined by

$$-w\frac{f'(w)}{f(w)} = \sum_{j=0}^{\infty} q_j w^j \tag{2.6.6}$$

and

$$E(w) = \sum_{k=0}^{\infty} E_k w^k. \tag{2.6.7}$$

It is clear, that (2.6.3) defines $\tilde{a}_k$ uniquely from $\tilde{a}_0, \ldots, \tilde{a}_{k-1}$, as long as $\varphi(k+r_2) \neq 0$. If we choose $\tilde{a}_0 = 1$, we get by (2.6.3), that all $\tilde{a}_k$ are polynomials in λ^{-2}. Therefore:

Theorem: *Let $r_1 > r_2$ be the two solutions to the indicial equation of (2.1.13). The following statements are equivalent*

1. *The equation (2.1.13) has a solution of the form (2.6.2).*
2. *(2.1.13) has two linearly independent meromorphic solutions.*
3. $$\sum_{s=0}^{r_1-r_2-1}(s+r_2)q_{r_1-r_2-s}\tilde{a}_s = \lambda^{-2}\sum_{s=2}^{r_1-r_2} E_{s-2}\tilde{a}_{r_1-r_2-s}, \tag{2.6.8}$$

   where $-w\frac{f'(w)}{f(w)} = \sum_{j=0}^{\infty} q_j w^j$ *and* $E(w) = \sum_{k=0}^{\infty} E_k w^k$.

Moreover, $r_1 = 0$, $r_2 = -n+1$, $q_0 = n$, if $z = z_0$ is a pole of f of order n, and $r_1 = n+1$, $r_2 = 0$, $q_0 = -n$, if $z = z_0$ is a zero of f of order n.

Remark: Condition (2.6.8) above is independent of the special shape of the function f. In sections 2.9–2.15 we will further investigate it for the local normalizations $f(z) = (z - z_0)^n$ and $f(z) = (z - z_0)^{-n}$. In these cases it is possible to derive a condition on the functions f and E, which is more explicit and computationally easier to handle.

We also note that in the case where f has a zero at $z = z_0$ the two linearly independent solutions are actually locally holomorphic.

Finally we get the following nice result for poles of second order:

Corollary: *If f has a pole of second order at $z = z_0$ with vanishing residue, then (2.1.1) has a locally meromorphic solution at $z = z_0$.*

Proof: Condition (2.6.8) in this case reduces to $q_1 = -\mathrm{res}_{z_0} f = 0$. □

2.7 At this point we are able to provide the

Proof of Theorem 2.2: The existence of a global solution to (2.1.1), (2.1.2) implies the existence of linearly independent local meromorphic solutions of (2.1.13), which are given by the entries in the right column of the matrix solution $g_-(z, \lambda)$. Because, if these entries are multiples of each other, then by (2.1.8) and (2.1.10) the rows of g_- are linearly dependent, which contradicts $\det g_-(z, \lambda) = 1$.

To prove the converse statement we first look at the local problem: We therefore look at a neighbourhood V of an arbitrary point z_0, where f has at most one pole or one zero, which is located at $z = z_0$.

Let us first consider the case where f has a zero of order n at z_0. Denote by y_1 and y_2 the two solutions of (2.1.13) associated with $r_1 = n+1$ and $r_2 = 0$, respectively. We have seen in section 2.6 that y_1 and y_2 are functions of λ^{-2}. Since, by the remark after equation (2.2.1), $y_1, y_2 \in \mathcal{A}$ as functions of λ, we can write

$$y_1 = a_0 + \lambda^{-2} a_2 + \dots, \tag{2.7.1}$$

$$y_2 = b_0 + \lambda^{-2} b_2 + \dots \tag{2.7.2}$$

From (2.1.13) we obtain that a_0 and b_0 satisfy the differential equation

$$q'' - \frac{f'}{f} q' = 0. \tag{2.7.3}$$

This equation has the solution

$$q' = \alpha f, \qquad q = \alpha \int_{z_0}^{z} f(z') \mathrm{d}z' + \beta, \tag{2.7.4}$$

for some complex constants α, β. Since $y_1(z_0) = 0$ and $y_2(z_0) = 1$ for all λ we get $a_0(z_0) = 0$, $b_0(z_0) = 1$. Therefore

$$a_0(z) = \alpha \int_{z_0}^{z} f(z') \mathrm{d}z', \qquad b_0 = 1 + \gamma \int_{z_0}^{z} f(z') \mathrm{d}z', \tag{2.7.5}$$

for some $\alpha, \gamma \in \mathbb{C}$. Thus we can find $\sigma, \tau \in \mathbb{C}$, $\tau \neq 0$, s.t. $u = \sigma y_1 + \tau y_2$ is of the form

$$u = 1 + u_2 \lambda^{-2} + \dots \tag{2.7.6}$$

Then u and y_1 are linearly independent and the matrix

$$R = \begin{pmatrix} \rho\lambda \frac{y_1'}{f} & \rho y_1 \\ \lambda \frac{u'}{f} & u \end{pmatrix}, \ \rho \in \mathbb{C} \tag{2.7.7}$$

satisfies (2.1.1). Here we have to choose $\rho = \rho(\lambda)$ in such a way, that R has coefficients in the Banach algebra $\mathcal{A}$. This is possible by the remark after (2.2.1), since the upper left entry of R is a solution of equation (2.1.14). In addition we have that

$$\det R = \rho\lambda f^{-1}(y_1' u - y_1 u'). \tag{2.7.8}$$

is independent of z and an element of $\mathcal{A}$. Since, up to a factor, it is equal to the Wronskian of (2.1.13) w.r.t. the linearly independent solutions y_1 and u, it is nonzero for all λ. It follows, that R is invertible, and the function $\det R$ is invertible in $\mathcal{A}$. We can therefore choose ρ, such that $\det R = 1$ for all λ. This implies, that the upper left entry is of the form $1 + \lambda^{-2} c_2 + \dots$ as a function of λ.

We have shown, that at points in D, where f has a zero, there exists a local solution to (2.1.1) in $\Lambda_*^- \mathbf{SL}(2, \mathbb{C})_\sigma$.

In the case, where f has a pole of order n, g has a zero of order $n + m$ at z_0, where $m \geq 0$ is the zero order of E at z_0. Therefore, we can argue with (2.2.1) instead of (2.1.13) as in the previous case. As a consequence one obtains also in this case a solution to (2.1.1) in $\Lambda_*^- \mathbf{SL}(2, \mathbb{C})_\sigma$.

We now have shown, that around every point in D there exists a local solution of (2.1.1) which lies in the group $\Lambda_*^- \mathbf{SL}(2, \mathbb{C})_\sigma$. The latter is isomorphic to the nontwisted based negative loop group $\Lambda_*^- \mathbf{SL}(2, \mathbb{C})$ by the restriction of the following homomorphism from $\Lambda \mathbf{SL}(2, \mathbb{C})$ to $\Lambda_*^- \mathbf{SL}(2, \mathbb{C})$: $(n \in \mathbb{Z})$

$$\lambda^n \sigma_+ \longrightarrow \lambda^{2n+1} \sigma_+, \tag{2.7.9}$$

$$\lambda^n \sigma_- \longrightarrow \lambda^{2n-1} \sigma_-, \tag{2.7.10}$$

$$\lambda^n \sigma_3 \longrightarrow \lambda^{2n} \sigma_3. \tag{2.7.11}$$

The matrices

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.7.12}$$

are Chevalley generators of $\mathbf{SL}(2, \mathbb{C})$.

We choose an open cover $(U_i)_{i \in I}$ of D and solutions g_i of (2.1.1) in the sets U_i, which take values in $\Lambda_*^- \mathbf{SL}(2, \mathbb{C})_\sigma$. Let $\tilde{g}_i : D \to \Lambda_*^- \mathbf{SL}(2, \mathbb{C})$ denote the image of g_i under the automorphism given above. Since the set of poles and zeroes of the meromorphic function $f(z)$ has no cluster points, we can assume that there is at most one such point in each U_i. Furthermore we may assume, that each $\tilde{g}_i$ is holomorphic in U_i, except possibly for

one pole which is a pole or zero of $f(z)$. By the definition of the negative loop group and the remark after equation (2.2.1), we know that each $\tilde{g}_i$ as a function of λ is analytic in $D_\infty = \{\lambda \in \mathbb{C} \,|\, |\lambda| > \frac{1}{2}\} \cup \infty$.

Next we consider functions $h_{ij} = \tilde{g}_i \tilde{g}_j^{-1}$, which are defined on the intersections $U_i \cap U_j$. These functions are independent of z, since

$$\mathrm{d}h_{ij} = (\mathrm{d}\tilde{g}_i)\tilde{g}_j^{-1} - \tilde{g}_i\tilde{g}_j^{-1}(\mathrm{d}\tilde{g}_j)\tilde{g}_j^{-1} = \tilde{g}_i\xi\tilde{g}_j^{-1} - \tilde{g}_i\tilde{g}_j^{-1}\tilde{g}_j\xi\tilde{g}_j^{-1} = 0,$$

and they satisfy the cocycle condition $h_{ij}h_{jk} = h_{ik}$.

We set $V_i = D_\infty$, $i \in I$, then $(V_i)_{i\in I}$ is an open cover of D_∞. Moreover, the h_{ij} form a cocycle relativ to the V_i. They define a rank 2 vector bundle over the noncompact Riemann surface D_∞. But any vector bundle over D_∞ is trivial. Therefore there exist functions $h_i(\lambda)$ on V_i taking values in $\mathbf{SL}(2,\mathbb{C})$, such that $h_i h_j^{-1} = \tilde{g}_i\tilde{g}_j^{-1}$. We may also assume that $h_i(\lambda) \to I$ for $\lambda \to \infty$. The relation $h_i^{-1}\tilde{g}_i = h_j^{-1}\tilde{g}_j$ defines a globally meromorphic function $\tilde{g}(z,\lambda)$ on $z \in D$, which is holomorphic for $\lambda \in D_\infty$ at each z, where it is finite. This shows that $\tilde{g}(z,\lambda)$ takes values in the based negative loop group $\Lambda_*^-\mathbf{SL}(2,\mathbb{C})$, which is isomorphic to the twisted group $\Lambda_*^-\mathbf{SL}(2,\mathbb{C})_\sigma$. Since ξ has no pole at $z = 0$, this global solution is defined and invertible at $z = 0$. It can therefore be chosen to satisfy (2.1.2). □

Corollary: *If (2.4.5) is satisfied for all points, where f has a pole or a zero, then there exist two globally linearly independent solutions of equation (2.1.13).*

<u>Proof:</u> If condition (2.4.5) is satisfied at every pole or zero of f, then locally around every point in D, there exist two linearly independent solutions to (2.1.13). Then Theorem (2.2) implies the existence of two globally linearly independent meromorphic solutions. □

2.8 In this section we collect the results obtained in the last sections.

Theorem 1:

a) (2.1.13) has always a meromorphic solution.

b) Let y_1 denote a meromorphic solution of (2.1.13). Then $y_2 = y_1 \cdot v$ is a solution of (2.1.13) if and only if

$$v' = C\frac{f}{y_1^2},\ C = \text{const.}$$

c) The equation (2.1.13) has two linearly independent solutions in the neighbourhood of a point $z = z_0$, if and only if

$$\oint_{z_0} \frac{f}{y_1^2}\mathrm{d}z = \mathrm{res}_{z=z_0}\frac{f}{y_1^2} = 0.$$

d) If (2.4.5) is satisfied for z_0, then the linearly independent solutions y_1 and y_2 can be chosen in the following way:

If f has a zero of order n at $z = z_0$, then both, y_1 and y_2 are locally holomorphic, y_1 has no zero at z_0 and y_2 has a zero of order $n+1$ at $z = z_0$.

If f has a pole of order n, then one solution is again locally holomorphic without a zero at $z = z_0$ and the other solution has a pole of order $n-1$ there.

As pointed out in section 2.5, equation (2.4.5) is invariant under coordinate changes, therefore Theorem 2.2 implies

Theorem 2: *Equation (2.1.1) has a (globally) meromorphic solution $g_-(z,\lambda)$ in the twisted loop group $\Lambda_*^-\mathbf{SL}(2,\mathbb{C})_\sigma$, if and only if equation (2.4.5) is satisfied at every pole or zero of $f(z)$.*

There is one case, in which condition (2.4.5) is automatically satisfied at a pole or zero z_0 of f.

Theorem 3: *If f and E are symmetric in $z - z_0$, then (2.1.1) has a local meromorphic solution around $z = z_0$ regardless whether f has a pole or a zero there.*

Proof: We set $w = z - z_0$. If f and E are symmetric in w, then with $y_1(w)$ also $\tilde{y}(w) = y_1(-w)$ is a solution of (2.1.13). Let y_1 be the meromorphic solution to the higher root r_1 of the indicial equation (which always exists and is actually holomorphic around $z = z_0$). We know that $\tilde{y}(z) = \alpha y_1(w) + \beta y_2(w)$, where $y_i(w)$ is a solution to (2.1.13) which belongs to the root r_i of the indicial equation, and α, β are independent of w.

We will use the discussion in sections 2.4 and 2.5. First let f have a pole at $z = z_0$, then y_1 is of the form (2.4.6). Therefore,

$$\tilde{y}(w) = 1 + \sum_{i=1}^{\infty} (-1)^i a_i w^i$$

has neither a pole nor a zero at $z = z_0$. This together with the fact that $\tilde{y}(0) = y_1(0) = 1$ implies, that $\tilde{y}(w) = y_1(w)$.

If f has a zero of even order n at $z = z_0$, then $y_1(w)$ is of the form (2.5.3). Therefore,

$$\tilde{y}(w) = -w^{n+1} + \sum_{i=1}^{\infty} (-1)^i a_i w^{i+n+1}$$

has a zero of the same order as y_1 at $z = z_0$. This together with the fact that the coefficient of w^{n+1} in $\tilde{y}$ is -1 implies that $\tilde{y}(w) = -y_1(w)$

Thus we have $\tilde{y}^2(w) = y_1^2(w)$ in both cases and, with y_1^2 and f being symmetric, $\frac{f}{y_1^2}$ cannot have a residue at $z = z_0$. □

Finally, this theorem is accompanied by a result similar to Corollary 2.6:

Corollary: *Let ξ be as in equation (1.1.1). In the following two cases (2.1.1) has locally a meromorphic solution around $z = z_0$:*

a) f has a pole at $z = z_0$ with even principal part and $f \cdot E$ has at most a pole of second order there.

b) f has a zero at $z = z_0$ and $f^{-1}E$ has no pole at $z = z_0$.

Proof: In Case b) neither f nor $f^{-1}E$ has a pole at $z = z_0$. Therefore the meromorphic potential is holomorphic around z_0. Around a point where the meromorphic potential is nonsingular, (2.1.1) has locally a meromorphic solution.

In Case a) $f \cdot E$ having at most a pole of second order implies, that E has at least a zero of order $n-2$. But then $E_0 = \ldots = E_{n-3} = 0$ and the condition (2.6.8) is a condition on the principal part of f only. With Theorem 2.8.3 we then get the result. $\square$

2.9 In the following sections we will further investigate the condition (2.6.8) in the special cases $f(z) = (z - z_0)^n$ and $f(z) = (z - z_0)^{-n}$.

2.10 Case I: f has a pole of order $n \geq 2$ at $z = z_0$.
We want to find a solution to (2.1.13) of the form

$$y_2 = \sum_{i \geq 0} \tilde{a}_i (z - z_0)^{i-n+1}, \; \tilde{a}_0 \neq 0. \tag{2.10.1}$$

In this section we will normalize $f = (z - z_0)^{-n}$, $n \geq 2$. This is possible without loss of generality because of the remark in section 2.1 about the invariance of (2.1.1) and (2.1.2) under coordinate changes.

In this case, with $\frac{f'}{f} = -\frac{n}{z-z_0}$, the recursion relation (2.6.3) reads

$$\tilde{a}_1 \;\; = \;\; 0, \tag{2.10.2}$$

$$k(k-n+1)\tilde{a}_k \;\; = \;\; \lambda^{-2} \sum_{i=2}^{k} E_{i-2} \tilde{a}_{k-i}, \; k \geq 2. \tag{2.10.3}$$

If we choose $\tilde{a}_0$ to be independent of λ, then, as was already noted in section 2.6, that the coefficients $\tilde{a}_k$ are even polynomials in λ^{-1}. In particular in this case $\deg \tilde{a}_k \leq k$ if k is even, $\deg \tilde{a}_k \leq k - 1$ if k is odd.

We also have the nontrivial condition (2.6.8) on E, which in this case becomes:

$$\sum_{i=2}^{n-1} \tilde{a}_{n-i-1} E_{i-2} = 0. \tag{2.10.4}$$

Here the E_i are the Taylor coefficients of $E(z)$:

$$E(z) = \sum_{k=0}^{\infty} E_k (z - z_0)^k.$$

Clearly, $\tilde{a}_0$ and $\tilde{a}_{n-1}$ can be chosen arbitrarily. If $\tilde{a}_0 = 0$, then the corresponding solution is $\tilde{a}_{n-1} y_2$, where y_2 denotes a holomorphic solution of (2.1.13).

For $n = 2$ the condition (2.10.4) is trivially satisfied. Therefore we always find a meromorphic matrix solution of (2.1.1) in this case.

2.11 Let now $n \geq 4$ and set $\tilde{a}_0 = 1$. Then (2.10.4) has a solution of the form

$$(1, 0, \tilde{a}_2, \ldots, \tilde{a}_{n-2}).$$

This equation involves all coefficients $E_0, E_1, \ldots, E_{n-3}$ except E_{n-4}, which can be chosen arbitrarily.

The left hand side of (2.10.4) is an even polynomial in λ^{-1} of degree at most $n - 3$. Its constant term is E_{n-3}, therefore

$$E_{n-3} = 0. \tag{2.11.1}$$

This implies, that if f has a pole of order $n \geq 4$ at $z = z_0$, and E has a zero of order $n-3$ there, then there will not be a meromorphic solution to (2.1.1).

In addition we have

Lemma: *Let* $\deg \tilde{a}_k$ *denote the degree of* $\tilde{a}_k$ *as a polynomial of* λ^{-1}*. Then for* $0 \leq k \leq n-2$

$$\deg \tilde{a}_k = \begin{cases} k, \text{ for } k \text{ even}, \\ k-1, \text{ for } k \text{ odd}. \end{cases} \tag{2.11.2}$$

If $k \geq 2$*, then the coefficient* c_k *of the highest* λ^{-1} *power in* $\tilde{a}_k$ *is a positive real multiple of* $(-1)^{\frac{k-1}{2}} E_1 E_0^{\frac{k-3}{2}}$*, if* k *is odd, and a positive real multiple of* $(-1)^{\frac{k}{2}} E_0^{\frac{k}{2}}$ *if* k *is even.*

Proof: Proof by induction. For $\tilde{a}_0 = 1$, $\tilde{a}_1 = 0$ equation (2.11.2) holds. If $n \geq 4$, then the lemma holds for

$$\tilde{a}_2 = \lambda^{-2}(6-2n)^{-1} E_0, \tag{2.11.3}$$

and if $n \geq 6$ then it holds also for

$$\tilde{a}_3 = \lambda^{-2}(8-2n)^{-1} E_1. \tag{2.11.4}$$

We assume that we have proved the lemma for $k \leq s-1$. Then by the recursion relation (2.10.2) we have that

$$\deg \tilde{a}_s \leq \deg \tilde{a}_{s-2} + 2 = s, \tag{2.11.5}$$

if s is even, and

$$\deg \tilde{a}_s \leq \deg \tilde{a}_{s-2} + 2 = \deg \tilde{a}_{s-3} + 2 = s-1, \tag{2.11.6}$$

if s is odd. Also, if s is even, the λ^{-s} coefficient c_s of $\tilde{a}_s$ is a negative real multiple of $E_0 c_{s-2} = \alpha(-1)^{\frac{s-2}{2}} E_0^{\frac{s-2}{2}}$, $\alpha \in \mathbb{R}^+$, by the induction assumption. If s is odd, the λ^{-s+1} coefficient c_s of $\tilde{a}_s$ is a negative real multiple of

$$E_0 c_{s-2} + E_1 c_{s-3} = (\alpha + \beta)(-1)^{\frac{s-3}{2}} E_1 E_0^{\frac{s-3}{2}}, \ \alpha, \beta \in \mathbb{R}^+, \tag{2.11.7}$$

by the induction assumption. Therefore the lemma follows also for $k = s$. □

Corollary: *The highest nonvanishing power of* λ^{-1} *in equation (2.10.4) is* $c(z)\lambda^{-n+4}$*, where* $c(z)$ *is the* λ^{-n+4} *coefficient of* $\tilde{a}_{n-3}E_0 + \tilde{a}_{n-4}E_1$*. The coefficient* $c(z)$ *is a nonzero real multiple of* $E_1 E_0^{\frac{n-4}{2}}$*.*

Proof: From lemma 2.11 and n even it follows, that the highest λ^{-1} power occuring on the l.h.s. of (2.10.4) is given by $\deg \tilde{a}_{n-3} = \deg \tilde{a}_{n-4} = n-4$. With the notation of lemma 2.11, the coefficient of λ^{-n+4} is $c_{n-3}E_0 + c_{n-4}E_1$. Since $c_{n-3} = \alpha E_1 E_0^{\frac{n-6}{2}}$ and $c_{n-4} = \beta E_0^{\frac{n-4}{2}}$ and α, β have the same sign, we get that $c(z)$ is a nonvanishing real multiple of $E_1 E_0^{\frac{n-4}{2}}$. □

This implies that, if f has a pole of order $n \geq 4$ at z_0, and $f(z) = (z-z_0)^{-n}$, then

$$E_1 E_0^{\frac{n-4}{2}} = 0.$$

2.12 We summarize the discussion above for $n = 2, 4, 6$.

$n = 2$: (2.10.4) is trivial, therefore (2.1.1) always has a meromorphic solution in a neighbourhood of a second order pole of f.

$n = 4$: (2.10.4) $\Leftrightarrow E_1 = 0$.

$n = 6$: (2.10.4) $\Leftrightarrow E_3 = E_0 E_1 = 0$. E_2 can be chosen arbitrarily.

2.13 Let us write the recursion relation

$$-\lambda^2 k(k-n+1)\tilde{a}_k + \sum_{i=2}^{k} E_{i-2}\tilde{a}_{k-i} = 0 \tag{2.13.1}$$

as a matrix relation

$$0 = Aa,\ a = (\tilde{a}_0, 0, \tilde{a}_2, \ldots)^\top, \tag{2.13.2}$$

where

$$A = \begin{pmatrix} 0 & & & & \\ 0 & 0 & & & \\ E_0 & 0 & \alpha_2 & & \text{\Large 0} \\ E_1 & E_0 & 0 & \alpha_3 & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix},\ \alpha_k = -\lambda^2 k(k-n+1). \tag{2.13.3}$$

We know $\tilde{a}_1 = 0$, therefore the second column of A is irrelevant. Also $\tilde{a}_0 = 1$, therefore we get

$$0 = \begin{pmatrix} E_0 \\ E_1 \\ \vdots \end{pmatrix} + \hat{A}\hat{a},\ \hat{a} = (\tilde{a}_2, \ldots)^\top \tag{2.13.4}$$

where $\hat{A}$ is the infinite matrix

$$\hat{A} = \begin{pmatrix} \alpha_2 & & & & \\ 0 & \alpha_3 & & & \\ E_0 & 0 & \alpha_4 & & \text{\Large 0} \\ E_1 & E_0 & 0 & \alpha_5 & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}. \tag{2.13.5}$$

Therefore

$$\begin{pmatrix} E_0 \\ E_1 \\ \vdots \\ E_{n-3} \\ \vdots \end{pmatrix} = -\begin{pmatrix} \alpha_2 & & & & \\ 0 & \alpha_3 & & & \\ \vdots & & \ddots & & \text{\Large 0} \\ E_{n-5} & \ldots & 0 & \alpha_{n-1} & \\ \vdots & & & & \ddots \end{pmatrix} \hat{a}. \tag{2.13.6}$$

Because of $\alpha_{n-1} = 0$, $\hat{A}$ has the structure

$$\hat{A} = \begin{pmatrix} & B & & & & & \\ E_{n-5} & \ldots & E_0 & 0 & 0 & & \\ \vdots & & \vdots & E_0 & 0 & \alpha_n & \text{\Large 0} \\ & & & & & & \ddots \end{pmatrix}. \tag{2.13.7}$$

where

$$B = \begin{pmatrix} \alpha_2 & & & & \\ 0 & \ddots & & & \\ E_0 & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \\ E_{n-6} & \dots & E_0 & 0 & \alpha_{n-2} \end{pmatrix} \tag{2.13.8}$$

is an invertible $(n-3)\times(n-3)$ matrix with $\det B = \prod_{k=2}^{n-2}\alpha_k \neq 0$. Moreover, B involves only $E_0, \dots, E_{n-4}$.

It follows

$$(\tilde{a}_2, \dots, \tilde{a}_{n-2})^\top = -B^{-1}(E_0, \dots, E_{n-4})^\top. \tag{2.13.9}$$

This is a (fairly) explicit formula for $\tilde{a}_2, \dots, \tilde{a}_{n-2}$. Substituting this into (2.10.4) we obtain

$$E_{n-3} = (E_0, E_1, \dots, E_{n-4}) S_u P B^{-1} (E_0, E_1, \dots, E_{n-4})^\top = 0 \tag{2.13.10}$$

where S_u is the shift $(x_1, \dots, x_{n-3})^\top \mapsto (x_2, \dots, x_{n-3}, 0)^\top$ and P is the permutation $(x_1, \dots, x_{n-3})^\top \mapsto (x_{n-3}, x_{n-4}, \dots, x_1)^\top$. Setting $Q = S_u P B^{-1}$ we split Q into a symmetric and an antisymmetric part $Q = Q_S + Q_A$. So we get for $\underline{E} = (E_0, E_1, \dots, E_{n-4})^\top$ the equation

$$\langle \underline{E}, Q\underline{E} \rangle = \langle \underline{E}, Q_S \underline{E} \rangle = 0. \tag{2.13.11}$$

2.14 Case II: f has a zero of order $n \geq 2$ at $z = z_0$.

We want to find a solution to (2.1.13) of the form

$$y_2 = \sum_{i \geq 0} \tilde{a}_i (z - z_0)^i, \; \tilde{a}_0 \neq 0. \tag{2.14.1}$$

Again, w.l.o.g. we normalize in the following $f = (z - z_0)^n$, $n > 0$. Equation (2.6.3) in this case reads

$$\tilde{a}_1 = 0, \tag{2.14.2}$$

$$k(k-n-1)\tilde{a}_k = \lambda^{-2} \sum_{i=2}^{k} E_{i-2} \tilde{a}_{k-i}, \; k \geq 2. \tag{2.14.3}$$

In addition, the coefficients of the holomorphic function E must satisfy condition (2.6.8) which takes the form

$$\sum_{i=2}^{n+1} \tilde{a}_{n-i+1} E_{i-2} = 0. \tag{2.14.4}$$

$\tilde{a}_0$ and $\tilde{a}_{n+1}$ can be chosen arbitrarily. As in Case I, for $\tilde{a}_0 = 0$ we get the always existing meromorphic solution y_1 with a zero of order $n+1$ at z_0.

We set $\tilde{a}_0 = 1$. Then (2.14.4) has a solution of the form $(1, 0, \tilde{a}_2, \dots, \tilde{a}_n)$. The coefficients $\tilde{a}_k$ are again even polynomials in λ^{-1}. We have $\deg \alpha_k \leq k$, if k is even, and $\deg \alpha_k \leq k-1$, if k is odd.

Thus equation (2.14.4) gives a condition on the first $n-1$ terms of the Taylor expansion of $E(z)$, with the exception of E_{n-2}, which can be chosen arbitrarily.

The left hand side of (2.14.4) is an even polynomial in λ^{-1} of degree at most $n-1$. Its constant term is E_{n-1}, therefore

$$E_{n-1} = 0, \tag{2.14.5}$$

This implies, that if $g = E \cdot f^{-1}$ has a pole of first order at $z = z_0$, then there is no meromorphic solution to (2.1.1).

By the same arguments as in the proof of lemma 2.11, we get

Lemma: *Let* $\deg \tilde{a}_k$ *denote the degree of* $\tilde{a}_k$ *as a polynomial of* λ^{-1}*. Then for* $0 \leq k \leq n$

$$\deg \tilde{a}_k = \begin{cases} k, \text{ for } k \text{ even}, \\ k-1, \text{ for } k \text{ odd}. \end{cases} \tag{2.14.6}$$

If $k \geq 2$*, then the coefficient* c_k *of the highest* λ^{-1} *power in* $\tilde{a}_k$ *is a positive real multiple of* $(-1)^{\frac{k-1}{2}} E_1 E_0^{\frac{k-3}{2}}$*, if* k *is odd, and a positive real multiple of* $(-1)^{\frac{k}{2}} E_0^{\frac{k}{2}}$*, if* k *is even.*

Again it follows like in the last section

Corollary: *The highest nonvanishing power of* λ^{-1} *in equation (2.14.4) is* $c(z)\lambda^{-n+4}$*, where* $c(z)$ *is the* λ^{-n+2} *coefficient of* $\tilde{a}_{n-1}E_0 + \tilde{a}_{n-2}E_1$*. The coefficient* $c(z)$ *is a nonvanishing real multiple of* $E_1 E_0^{\frac{n-2}{2}}$*.*

This implies that, if f has a zero of order $n \geq 2$ at z_0, $f(z) = (z - z_0)^n$, then

$$E_1 E_0^{\frac{n-2}{2}} = 0.$$

2.15 In the cases $n = 2$ and $n = 4$ these results can be summarized as follows

$n = 2$: (2.14.4) $\Leftrightarrow E_1 = 0$.

$n = 4$: (2.14.4) $\Leftrightarrow E_3 = E_0 E_1 = 0$. E_2 can be chosen arbitrarily.

We write the recursion relation

$$-\lambda^2 k(k-n-1)\tilde{a}_k + \sum_{i=2}^{k} E_{i-2}\tilde{a}_{k-i} = 0 \tag{2.15.1}$$

as a matrix relation

$$0 = Aa, \; a = (\tilde{a}_0, 0, \tilde{a}_2, \ldots)^\top, \tag{2.15.2}$$

where

$$A = \begin{pmatrix} 0 & & & & \\ 0 & 0 & & & \\ E_0 & 0 & \alpha_2 & & \Large 0 \\ E_1 & E_0 & 0 & \alpha_3 & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}, \; \alpha_k = -\lambda^2 k(k-n-1). \tag{2.15.3}$$

We have $\tilde{a}_0 = 1$, $\tilde{a}_1 = 0$, $\alpha_{n+1} = 0$, and by proceeding as in Case I we get

$$(\tilde{a}_2, \ldots, \tilde{a}_n)^\top = -B^{-1}(E_0, \ldots, E_{n-2})^\top. \tag{2.15.4}$$

where B is the $(n-1)\times(n-1)$ matrix

$$B = \begin{pmatrix} \alpha_2 & & & & \\ 0 & \alpha_3 & & & \\ E_0 & 0 & \ddots & & 0 \\ \vdots & & & \ddots & \\ E_{n-4} & E_{n-5} & \dots & 0 & \alpha_n \end{pmatrix} . \tag{2.15.5}$$

We note that B is invertible with $\det B = \prod_{j=2}^{n} \alpha_j \neq 0$. Moreover B involves only $E_0, \dots, E_{n-2}$.

Condition (2.14.4) reads

$$E_{n-1} = (E_0, E_1, \dots, E_{n-2}) S_u P B^{-1} (E_0, E_1, \dots, E_{n-2})^\top = 0 \tag{2.15.6}$$

where S_u is the shift $(x_1, \dots, x_{n-1})^\top \mapsto (x_2, \dots, x_{n-1}, 0)^\top$ and P is the permutation $(x_1, \dots, x_{n-1})^\top \mapsto (x_{n-1}, x_{n-2}, \dots, x_1)^\top$. Setting $Q = S_u P B^{-1}$ we may split Q into a symmetric and an antisymmetric part $Q = Q_S + Q_A$. So we get for $\underline{E} = (E_0, E_1, \dots, E_{n-2})^\top$ the equation

$$\langle \underline{E}, Q\underline{E} \rangle = \langle \underline{E}, Q_S \underline{E} \rangle = 0. \tag{2.15.7}$$

3 Smoothness of the Iwasawa decomposition

3.1 In the following we will assume that we have found a meromorphic solution of (1.1.2)

$$\mathrm{d}g_- = g_- \xi,\ g_-(z,\lambda) \in \Lambda_*^- \mathbf{SL}(2,\mathbb{C})_\sigma,\ g_-(0,\lambda) = I. \tag{3.1.1}$$

We are now looking for conditions on ξ, s.t.

a) the orthogonal part F of an Iwasawa decomposition of g_-

$$g_-(z,\lambda) = F(z,\lambda) g_+^{-1}(z,\lambda) \tag{3.1.2}$$

is a smooth function in $z \in D$. Here $g_+ \in \Lambda^+ \mathbf{SL}(2,\mathbb{C})_\sigma$, $F \in \Lambda \mathbf{SU}(2)_\sigma$,

b) the map $\Phi : D \to \mathbb{R}^3$ defined by the Sym-Bobenko formula (1.1.4) is an immersion. It then automatically describes a CMC surface in $\mathbb{R}^3$ without branchpoints.

We would like to point out, that whenever ξ can be integrated to a meromorphic g_-, it is possible to split $g_-(z,\lambda)$ like in (3.1.2), where $F(z,\lambda) \in \Lambda \mathbf{SU}(2)_\sigma$, $g_+ \in \Lambda^+ \mathbf{SL}(2,\mathbb{C})_\sigma$. If g_- has a pole at $z = z_0$, then the Sym-Bobenko formula again defines a smooth CMC map on a deleted neighbourhood of z_0, but in general this map will have a singularity at z_0.

Our main tool to classify those meromorphic potentials which belong to CMC immersions, will be the dressing method.

We recall, that by normalization the coefficients f and $g = f^{-1}E$ of ξ are holomorphic in a neighbourhood of $z = 0$.

3.2 Let $\mathcal{M}$ denote the set of meromorphically integrable meromorphic potentials

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & f \\ \frac{E}{f} & 0 \end{pmatrix} \mathrm{d}z \tag{3.2.1}$$

on D. We also require, that for $\xi \in \mathcal{M}$ the coefficient f of ξ does not vanish identically, and that f has only poles and zeroes of even order. On $\mathcal{M}$ we want to define an action of the group $\Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$.

Let $\xi \in \mathcal{M}$ and denote by g_- the unique solution of (1.1.2). For $h_+ \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$ we consider h_+g_-. In general, h_+g_- will not be in the big cell $\Lambda_*^-\mathbf{SL}(2,\mathbb{C})_\sigma \cdot \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$. However, the argument in [5, §2] can be applied (almost) verbatim to our situation and we obtain

Proposition: *Let $\xi \in \mathcal{M}$, $h_+ \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$ and denote by g_- the solution of (1.1.2). Then there exists a discrete subset S of D and $\hat{g}_-(z,\lambda) \in \Lambda_*^-\mathbf{SL}(2,\mathbb{C})_\sigma$, $\hat{g}_+(z,\lambda) \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$, s.t.*

$$h_+(\lambda)g_-(z,\lambda) = \hat{g}_-(z,\lambda)\hat{g}_+(z,\lambda), \tag{3.2.2}$$

for all $z \in D \setminus S$. Moreover, $\hat{g}_-$ and $\hat{g}_+$ are meromorphic in D.

By differentiation we obtain $\hat{\xi} = \hat{g}_-^{-1}\mathrm{d}\hat{g}_-$. From the proposition above we know, that $\hat{\xi}$ is meromorphic in D and is of the form

$$\hat{\xi} = \lambda^{-1} \begin{pmatrix} 0 & \hat{f}(z) \\ \hat{g}(z) & 0 \end{pmatrix} \mathrm{d}z. \tag{3.2.3}$$

From the definition of $\hat{\xi}$ it is clear that $\hat{\xi} \in \mathcal{M}$ holds. We set

$$\hat{\xi} = h_+.\xi. \tag{3.2.4}$$

From (3.2.2) and $\hat{\xi} = \hat{g}_-^{-1}\mathrm{d}\hat{g}_-$ it follows

$$\hat{g}_-^{-1}\mathrm{d}\hat{g}_- = -\mathrm{d}\hat{g}_+ \cdot \hat{g}_+^{-1} + \hat{g}_+ g_-^{-1}\mathrm{d}g_- \cdot \hat{g}_+^{-1} \tag{3.2.5}$$

or

$$h_+.\xi = \hat{\xi} = -\mathrm{d}\hat{g}_+ \cdot \hat{g}_+^{-1} + \hat{g}_+\xi\hat{g}_+^{-1}. \tag{3.2.6}$$

It follows that $\hat{g} = \hat{f}^{-1}E$. As a consequence, the (possibly singular) CMC surfaces associated with ξ and $\hat{\xi}$ have the same Hopf differential.

From the uniqueness of the Riemann-Hilbert splitting it is easy to derive, that (3.2.6) defines a group action of $\Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$ on $\mathcal{M}$.

Finally we note

Theorem: *Let $\xi \in \mathcal{M}$ and $h_+ \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$. Then the CMC map associated with ξ is smooth near $z = z_0$ iff the CMC map associated with $h_+.\xi$ is smooth near $z = z_0$.*

Proof: Assume ξ leads to a smooth CMC map. Then $g_- = Fg_+^{-1}$ for some smooth $F \in \Lambda\mathbf{SU}(2)_\sigma$. Therefore $h_+g_- = h_+Fg_+^{-1} = \hat{g}_-\hat{g}_+ = \hat{F}\hat{p}_+$, where $\hat{F} \in \Lambda\mathbf{SU}(2)_\sigma$ and $p_+ \in$

$\Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$ are defined by the last equation. Clearly, $\hat{F}$ defines the CMC map associated with $\hat{\xi} = h_+.\xi$.

Moreover, $h_+F = \hat{F}\hat{p}_+g_+$. Comparing this to the Iwasawa splitting $h_+F = \tilde{F}\tilde{s}_+$ of h_+F we see that $\hat{F}$ is smooth—possibly up to a λ-independent diagonal factor $k(z,\bar{z})$, $\hat{F} = \tilde{F}k$. But the Sym-Bobenko formula yields the same immersion for $\hat{F}$ and $\tilde{F}$. This proves that the CMC map associated with $\hat{\xi} = h_+.\xi$ is also smooth. Furthermore, $\hat{\xi} = h_+^{-1}.\xi$, since (3.2.6) defines a group action. Therefore, the converse direction of the equivalence follows by the same arguments. □

Corollary 1: *The CMC map associated with $\xi \in \mathcal{M}$ is smooth near $z = z_0$ iff there exists some $h_+ \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$ such that the CMC map associated with $h_+.\xi$ is smooth near $z = z_0$.*

Proof: If the CMC map associated with ξ is smooth then we choose $h_+ = I$. On the other hand, if an h_+ exists, s.t. $h_+.\xi$ is associated with a smooth CMC map, then by Theorem 3.2 the CMC map associated to ξ is also smooth. □

Corollary 2: *Let $\xi \in \mathcal{M}$ and $h_1, \ldots, h_m \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$. Assume, that $h_1.(h_2.(\ldots h_m.\xi)\ldots)$ is holomorphic near $z = z_0$. Then the CMC map associated with ξ is smooth near $z = z_0$.*

Proof: The corollary follows from Theorem 3.2 and the fact, that the dressing (3.2.6) is a group action. □

3.3 In this section we will discuss some 'basic' dressing transformations. It is easy to verify, that the Lie algebra $\Lambda^+\mathbf{sl}(2,\mathbb{C})_\sigma$ is generated by the following three matrices:

$$U = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, V = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}. \tag{3.3.1}$$

Therefore, all dressing transformations are products (and limits) of dressing transformations associated with these three. We will therefore investigate what the three corresponding 'basic' transformations do.

We consider meromorphic potentials $\xi \in \mathcal{M}$ with defining functions f and E. What we will need and compute with are the coefficients of g_-. As before we write

$$g_- = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.3.2}$$

with

$$\begin{aligned} a &= 1 + \sum_{i\geq 1} a_i\lambda^{-i}, \\ b &= \sum_{i\geq 1} b_i\lambda^{-i}, \\ c &= \sum_{i\geq 1} c_i\lambda^{-i}, \\ d &= 1 + \sum_{i\geq 1} d_i\lambda^{-i}. \end{aligned} \tag{3.3.3}$$

The function $f(z)$ is obtained by differentiation from b_1 and also from c_1

$$f(z) = \frac{\mathrm{d}b_1(z)}{\mathrm{d}z} = E(z)\left(\frac{\mathrm{d}c_1(z)}{\mathrm{d}z}\right)^{-1}. \tag{3.3.4}$$

It is therefore sufficient to determine the behaviour of one of these coefficients under a dressing transformation.

Here is our general procedure: Let Q be one of the matrizes D, U or V. Set $H = \exp(tQ)$, $t \in \mathbb{C}$. Then by the definition of the dressing action we need to compute the Birkhoff splitting $Hg_- = \tilde{g}_-\tilde{g}_+$. We will do this in detail in the following sections.

We would like to point out here though, that in all three cases $\exp(tQ)$ is of particularly simple form:

$$\exp(tU) = \begin{pmatrix} 1 & 0 \\ t\lambda & 1 \end{pmatrix}, \tag{3.3.5}$$

$$\exp(tV) = \begin{pmatrix} 1 & t\lambda \\ 0 & 1 \end{pmatrix}, \tag{3.3.6}$$

$$\exp(tD) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}. \tag{3.3.7}$$

3.4 $Q = D$: We note $Hg_- = Hg_-H^{-1}\cdot H$ and in this case we simply have $\tilde{g}_- = Hg_-H^{-1}$, $\tilde{g}_+ = H$. Hence the λ^{-1} coefficient of the upper right corner of $\tilde{g}_-$ is given by

$$\tilde{b}_1 = e^{2t}b_1. \tag{3.4.1}$$

By differentiation we thus obtain

$$\tilde{f} = e^{2t}f. \tag{3.4.2}$$

I.e. the new b_1 and the new f are just constant multiples of the old ones. This has been used before and also been visualized in a beautiful movie by Charlie Gunn.

3.5 $Q = U$: We multiply the matrix Hg_- with a matrix of the form

$$G = \begin{pmatrix} 1 & 0 \\ q\lambda & 1 \end{pmatrix}, \tag{3.5.1}$$

where we choose q such that the λ term of Hg_-G vanishes. A straightforward computation using (3.3.2), (3.3.3) and (3.3.5) shows

$$Hg_-G = \lambda\begin{pmatrix} 0 & 0 \\ t + q(1+tb_1) & 0 \end{pmatrix} + \begin{pmatrix} 1+qb_1 & 0 \\ 0 & 1+tb_1 \end{pmatrix} + \lambda^{-1}\begin{pmatrix} 0 & b_1 \\ * & 0 \end{pmatrix}, \tag{3.5.2}$$

therefore $q = -\frac{t}{1+tb_1}$. This gives for the coefficient of λ^0 in Hg_-G

$$\tilde{g}_0 = \begin{pmatrix} \frac{1}{1+tb_1} & 0 \\ 0 & 1+tb_1 \end{pmatrix}. \tag{3.5.3}$$

Splitting off $\tilde{g}_0$ from Hg_-G we obtain $Hg_-G = p_-\tilde{g}_0$, $p_- \in \Lambda_*^-\mathbf{SL}(2,\mathbb{C})_\sigma$. Therefore, $\tilde{g}_+$ is the product of $\tilde{g}_0$ with G^{-1}:

$$\tilde{g}_+ = \begin{pmatrix} \frac{1}{1+tb_1} & 0 \\ t\lambda & 1+tb_1 \end{pmatrix} \tag{3.5.4}$$

and $\tilde{g}_- = Hg_-\tilde{g}_+^{-1} = Hg_-G\tilde{g}_0^{-1}$. The upper right corner $\tilde{b}_1$ of the λ^{-1}-coefficient of $\tilde{g}_-$ is therefore

$$\tilde{b}_1 = b_1(1+tb_1)^{-1}. \tag{3.5.5}$$

A differentiation using (3.3.4) yields

$$\tilde{f} = f(1+tb_1)^{-2}. \tag{3.5.6}$$

Remark: The formula (3.5.6) is very interesting, since, using the dressing generated by U, we move from a constant f this way to an $\tilde{f}$ with a quadratic pole. But this formula also shows, that we move away from a pole of order n with any arbitrarily small $t \neq 0$ and into a zero of degree $n-2$. We will exploit this in detail later.

3.6 $Q = V$: We multiply the matrix Hg_- with a matrix of the form

$$G = \begin{pmatrix} 1 & q\lambda \\ 0 & 1 \end{pmatrix}, \tag{3.6.1}$$

where we choose q such that the λ term of Hg_-G vanishes. Again, a straightforward computation using (3.3.2), (3.3.3) and (3.3.6) shows

$$Hg_-G = \lambda \begin{pmatrix} 0 & t+q(1+tc_1) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1+tc_1 & 0 \\ 0 & 1+qc_1 \end{pmatrix} + \lambda^{-1} \begin{pmatrix} 0 & * \\ c_1 & 0 \end{pmatrix}, \tag{3.6.2}$$

therefore $q = -\frac{t}{1+tc_1}$. This gives for the coefficient of λ^0 in Hg_-G

$$\tilde{g}_0 = \begin{pmatrix} 1+tc_1 & 0 \\ 0 & \frac{1}{1+tc_1} \end{pmatrix}. \tag{3.6.3}$$

We can again split off $\tilde{g}_0$ from Hg_-G and obtain $Hg_-G = p_-\tilde{g}_0$, $p_- \in \Lambda_*^-\mathbf{SL}(2,\mathbb{C})_\sigma$. Therefore, $\tilde{g}_+$ is again the product of $\tilde{g}_0$ with G^{-1}:

$$\tilde{g}_+ = \begin{pmatrix} 1+tc_1 & t\lambda \\ 0 & \frac{1}{1+tc_1} \end{pmatrix} \tag{3.6.4}$$

and $\tilde{g}_- = Hg_-\tilde{g}_+^{-1} = Hg_-G\tilde{g}_0^{-1}$. The lower left corner $\tilde{c}_1$ of the λ^{-1}-coefficient of $\tilde{g}_-$ is therefore

$$\tilde{c}_1 = c_1(1+tc_1)^{-1}. \tag{3.6.5}$$

A differentiation using (3.3.4) gives

$$\tilde{f} = f(1+tc_1)^2. \tag{3.6.6}$$

Remark: It is only possible to add new poles of f with this transformation if f has zeroes.

3.7 Writing the results above in a compact form, we obtain with obvious notation

$$\begin{aligned} T_D(t)f &= e^{2t}f, & (3.7.1)\\ T_U(t)f &= f(1+tb_1)^{-2}, & (3.7.2)\\ T_V(t)f &= f(1+tc_1)^2. & (3.7.3)\end{aligned}$$

By the discussion of the dressing transformations in section 3.2 we know, that every meromorphically integrable meromorphic potential ξ is transformed into another meromorphically integrable meromorphic potential $\hat{\xi}$. Therefore, if f is the coefficient of a $\xi \in \mathcal{M}$, the transformations (3.7.1)–(3.7.3) on the group level are really coming from a group action (3.2.6) of $\Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$. In our context it is therefore enough to investigate the effect of the basic dressing transformations by their action on the coefficient f.

3.8 In this section we discuss in some more detail the effect of T_U and T_V with regard to creating/annihilating poles or zeroes (note that T_D has no effect in this regard).

Consider $z_0 \in D$ such that

(1) f has a zero of order n and E has a zero of order m at z_0.

Expanding relative to $w = z - z_0$, we know

$$b_1 = \int f(z)\mathrm{d}z = \beta_0 + \beta_1 w^{n+1} + \ldots,\ \beta_1 \neq 0. \tag{3.8.1}$$

Therefore, the denominator of $T_U(t)f$ is

$$1 + tb_1 = (1 + t\beta_0) + t\beta_1 w^{n+1} + \ldots. \tag{3.8.2}$$

This implies that $1 + tb_1$ does not vanish at z_0 if t is small.

(1a) If t is sufficiently small, then $T_U(t)f$ has a zero of order n at z_0. If $\beta_0 \neq 0$ and $t = -\beta_0^{-1}$, then $T_U(t)f$ has a pole of order $n+2$ at z_0.

Next we consider $T_V(t)f$. Therefore we consider

$$c_1 = \int \frac{E(z)}{f(z)}\mathrm{d}z \tag{3.8.3}$$

If $m \geq n$, then $\frac{E(z)}{f(z)}$ is holomorphic at $z = z_0$ and has a zero at z_0 of order $m - n$. In this case

$$c_1 = \gamma_0 + \gamma_1 w^{m-n+1} + \ldots,\ \gamma_1 \neq 0, \tag{3.8.4}$$

and

$$1 + tc_1 = 1 + t\gamma_0 + t\gamma_1 w^{m-n+1} + \ldots \tag{3.8.5}$$

(1b) Assume $m \geq n$. If t is sufficiently small, then $T_V(t)f$ has a zero of order n at z_0. If $\gamma_0 \neq 0$ and $t = -\gamma_0^{-1}$, then $T_V(t)f$ has a zero of order $2m + 2 - n$ at z_0.

Assume now $m < n$. Then $\frac{E(z)}{f(z)}$ has a pole at z_0 of order $n - m$. In this case

$$c_1 = \gamma w^{-(n-m-1)} + \ldots,\ \gamma \neq 0. \tag{3.8.6}$$

As a consequence, in this case for all $t \neq 0$, c_1 has a pole of order $n - m - 1$ at z_0. This implies

(1c) Assume $m < n$. Then for all $0 \neq t \in \mathbb{C}$ the function $T_V(t)f$ has a zero of order $2m+2-n$, if $2m+2-n \geq 0$, or a pole of order $n-2m-2$, if $2m+2-n < 0$.

Remark: We note, that the case (1b) cannot occur if f and E are associated with a CMC immersion around z_0: If $m \geq n$, then f and $\frac{E}{f}$ are both holomorphic in a neighbourhood of z_0. But then also g_- is holomorphic there and the splitting $g_- = F g_+^{-1}$ produces smooth F and g_+ in a neighbourhood of z_0. As a consequence, the Sym-Bobenko formula shows (see section A.8), that Φ_z and $\Phi_{\bar{z}}$ vanish at z_0, i.e. Φ has a branchpoint there.

Next we consider the case

(2) f has a pole of order n and E has a zero of order m at z_0.

In this case

$$b_1 = \beta w^{-n+1} + \ldots, \; \beta \neq 0, \tag{3.8.7}$$

and

$$(1 + t b_1)^{-2} = (t\beta)^{-2} w^{2n-2} + \ldots \tag{3.8.8}$$

has a zero of order $2n-2$ at z_0, independent of the choice of $t \neq 0$.

(2a) For every $0 \neq t \in \mathbb{C}$, the function $T_U(t)f$ has a zero of order $n-2$ at z_0.

Finally we consider again $T_V(t)f$. Here we need to consider

$$c_1 = \int \frac{E(z)}{f(z)} \mathrm{d}z = \gamma_0 + \gamma_1 w^{m+n+1} + \ldots, \; \gamma_1 \neq 0, \tag{3.8.9}$$

$$1 + t c_1 = 1 + t\gamma_0 + t\gamma_1 w^{m+n+1} + \ldots \tag{3.8.10}$$

(2b) If t is sufficiently small, then $T_V(t)f$ has a pole of order n at z_0. If $\gamma_0 \neq 0$, and $t = -\gamma_0^{-1}$, then $T_V(t)f$ has a zero of order $2m+2+n$ at z_0.

3.9 In the following section we discuss what poles and zeroes one can generate and annihilate with dressing transformations. We recall that $\mathcal{M}$ denotes the set of meromorphic potentials, which can be integrated to meromorphic g_-. Also, if $\xi \in \mathcal{M}$, then f does not vanish identically (by convention).

Theorem: *Let $\xi \in \mathcal{M}$ and N be any positive integer. Then*

a) *For every open subset $U \subset D$ there exists an open subset $V_0 \subset U$, s.t. for every $z_0 \in V_0$ there exists an element $\tilde{\xi}$ in the dressing orbit of ξ, s.t. the coefficient $\tilde{f}$ has a pole of order $2N$ at z_0.*

b) *For every open subset $U \subset D$ there exists an open subset $V_1 \subset U$, s.t. for every $z_0 \in V_1$ there exists an element $\hat{\xi}$ in the dressing orbit of ξ, s.t. the coefficient $\hat{f}$ has a zero of order $2N$ at z_0.*

Proof: First we show how to create poles of f.
Let b_1 and c_1 be as above, and choose an arbitrary open subset $U_1 \subseteq U$, s.t. $b_1(z) \neq 0$ and $E(z) \neq 0$ for all $z \in U_1$. This is possible, since b_1 is meromorphic and not identically zero, and we exclude the case of the round sphere. For every $z \in U_1$ the dressing transformation $T_U(t_1)$, $t_1 = -b_1(z)^{-1}$, by (3.8.1a) applied to $n = 0$, produces a pole of order 2 at z. We now look at the new $c_1^{(1)}(z) = \int \frac{E(z)}{\tilde{f}(z)} \mathrm{d}z$, where $\tilde{f}$ is defined by (3.7.2). We choose a subset $U_2 \subseteq U_1$, s.t. $c_1^{(1)}$ has no zeroes in U_2. Again, this is possible, since $c_1^{(1)}$ is nontrivial and meromorphic. For each $z \in U_2$ the successive application of $T_U(b_1(z)^{-1})$ and $T_V(c_1^{(1)}(z)^{-1})$, by (3.8.2b) applied to $n = 2$, produces a zero of order 4 at z. Here we also use, that $E(z) \neq 0$ on U_1.

We continue with this method producing higher order poles and zeroes at every step. If necessary, we restrict ourselves to smaller open subsets U_n of U. For f this produces poles of order $4k - 2$, $k \in \mathbb{N}$ and zeroes of order $4k$. On the other hand, starting with the transformation T_V reverses the role of poles and zeroes. For f we get poles of order $4k$ and zeroes of order $4k - 2$, $k \in \mathbb{N}$. $\square$

Corollary: *Let $z_0 \in D$, $\epsilon > 0$ and N be a positive integer. Then there exist $z_1, z_2 \in D$, $\|z_1 - z_0\| < \epsilon$, $\|z_2 - z_0\| < \epsilon$ and dressing transformations T_1 and T_2, s.t. $f_1 = T_1 f$ has a pole of order $2N$ at z_1 and $f_2 = T_2 f$ has a zero of order $2N$ at z_2.*

3.10 In the last section we have seen, that by using dressing transformations we can create poles and zeroes of f. In the next sections, we are mainly interested in the question whether we can —at least locally— remove poles and zeroes of f.

For a $\xi \in \mathcal{M}$ we will say, that a pole (resp. zero) z_0 of f can be "dressed away locally", if there exists a dressing transformation T s.t. Tf is defined and nonzero at z_0. Similarly we define "dressing away locally" of poles and zeroes for $f^{-1}E$.

Theorem: *Let $\xi \in \mathcal{M}$ with coefficients f and $f^{-1}E$. If $E(z_0) \neq 0$, then every pole or zero of f at z_0 can be dressed away locally.*

Proof: Poles and zeroes of f are of even order. If E has no zeroes, then, by (3.8.1c), $T_V(t)$ produces a pole of order $n-2$ out of a zero of order $n \geq 4$ and $T_U(t)$, by (3.8.2a), produces a zero of order $n-2$ out of a pole of order $n \geq 4$. This holds for all $t \neq 0$. Finally a pole (zero) of order 2 is reduced by $T_U(t)$ ($T_V(t)$) to a point, where f is finite and different from zero. $\square$

3.11 We would like to generalize the theorem above to the case of a general E. But it seems, that even in view of $\xi \in \mathcal{M}$ this is not possible. However, we can prove

Theorem: *Let $\xi \in \mathcal{M}$ and m be the zero order of $E(z)$ at $z_0 \in D$.*

If f has a zero of order n at z_0, then this zero of f can be dressed away locally, if

$$r(2m+4) - m - 2 \leq n \leq r(2m+4) \tag{3.11.1}$$

for some integer $r \geq 1$.

If f has a pole of order n at z_0, then this pole can be dressed away locally, if $n = 2$ or

$$r(2m+4) - m \leq n \leq r(2m+4) + 2, \tag{3.11.2}$$

for some integer $r \geq 1$

<u>Proof:</u> We proceed as above. Poles of $f^{-1}E$ cannot be simple, since g_- is meromorphic. If E has a zero at z_0 of order m and f has a zero of order $n = 2k \geq m+2$ there, then $T_V(t)$, by (3.8.1c), will produce a pole of order $n-2m-2$ at z_0. $T_U(t)$, by (3.8.2a), produces a zero of order $n-2$ from a pole of order n. So if we start with an f which has a zero of order n and an E with a zero of order m, then there are the following cases:

1. $n-2m-2 \leq 0$: We are done. We get a zero of order $2+2m-n \geq 0$ of f which gives a zero of order $n-m-2 \geq 0$ of Ef^{-1} (with a zero or pole of order 0 we always mean a point where f is finite and nonvanishing).

2. $n-2m-2 > 0$: In this case we have a pole of order $n-2m-2$ after transforming with $T_V(t)$. By (3.8.2a), this pole is reduced to a zero of order $n-2m-4$ after transforming with $T_U(t)$. If $n-2m-4 = 0$, we are done. Otherwise we repeat the whole procedure with $n-2m-4$ instead of n. Notice, that in this case, if n satisfies condition (3.11.1) for some integer r, then $r \geq 2$. Therefore $n-(2m+4)$ satifies the condition for $r-1 \geq 1$.

If we start with a pole of f of order n, we apply first $T_U(t)$. If $n = 2$, then we get by (3.8.2a) a locally holomorphic $\tilde{f} = T_U(t)f$ without zero at z_0. If $n \geq 4$, we get a zero of $\tilde{f} = T_U(t)f$ of order $n-2$. If the pole order n satisfies (3.11.2) for some integer $r \geq 1$, then the zero order $n-2$ satisfies (3.11.1) for the same integer r. This case can then be treated like the case of a zero of f. $\square$.

Remark: It is important that in the process of pole/zero order reduction there is no Ef^{-1} which has a pole of first order, because this is impossible for the coefficient of a meromorphically integrable potential. We get the following result:

Corollary: *Let ξ be of the form (1.1.1) and let m be the zero order of E at a point $z_0 \in D$. If f has a zero of order n at z_0 and*

$$n = k(m+2) - 1$$

for some positive odd integer k, then $\xi \notin \mathcal{M}$. Similarly, if f has a pole of order n at z_0 and

$$n = k(m+2) + 1$$

for some positive odd integer k, then $\xi \notin \mathcal{M}$.

<u>Proof:</u> In the first case, if $k = 2r+1$, then $n - r(2m+4) - m = 1$. Therefore the procedure for pole/zero order reduction described above leads to an Ef^{-1} in the dressing orbit with a simple pole. The second part reduces to the first one after applying a dressing transformation $T_U(t)$ to f. $\square$

It is clear, that the condition of the corollary can only be satisfied by an m which is odd, since n is always even.

3.12 In the last section we did not exclude the case of a CMC map with branchpoints. If we exclude branchpoints, i.e. if we restrict ourselves really to CMC immersions over the

domain D, then we get much more restrictive conditions on the pole and zero orders of the functions $f(z)$ and $E(z)$.

As above let f and $f^{-1}E$ be the entries of a meromorphic potential, which describes a CMC map without branchpoints.

This additional feature of ξ will be expressed by the fact, that if f has a zero at a point z_0 then $g = f^{-1}E$ must have a pole there, otherwise we get a branchpoint of the CMC map (see Theorem A.8).

We will proceed similar to 3.11. We will apply alternatingly $T_U(t)$ and $T_V(s)$ until we have reached a potential $\hat{\xi}$, which is holomorphic at z_0, where f had originally a pole or a zero.

As pointed out in 3.1, the CMC map associated with $\hat{\xi}$ extends to a smooth immersion across z_0. Therefore, by Theorem 3.2, the CMC map associated with the original potential ξ also extends to a smooth immersion across z_0.

Let us start with an f that has a pole of order $n > 0$ and an E that has a zero of order m at $z = z_0$.

If we use $T_U(t)$ on f, by (3.8.2a) we get an f with a zero of order $n-2$ and a g with a zero of order $m+2-n$ if $n \leq m+2$ or a pole of order $n-m-2$ if $n > m+2$, since also for the transformed functions we have $\hat{g} = \hat{f}^{-1}E$ by the remark after equation (3.2.6). If $n-2 > 0$ then the new meromorphic potential belongs to a CMC immersion only if $n > m+2$. If $n-2 = 0$ then we have found a locally holomorphic potential in the dressing orbit of the original meromorphic potential.

Otherwise we apply $T_V(t)$ to f. Since $m < n-2$, (3.8.1c) applies, and yields a new f with a zero of order $2m+4-n$ at z_0, if $2m+4-n \geq 0$, or a pole of order $n-2m-4$ at z_0, if $2m+4-n < 0$.

If $2m+4 = n$, then $f(z_0) \neq 0$. In this case g has a zero of order m at z_0. This means, that we arrived at a holomorphic potential.

If $2m+4 > n$, then f has a zero of order $2m+4-n$ at z_0, therefore g must have a pole at z_0, i.e. $m-(2m+4-n) = n-m-4 < 0$.

If $2m+4 < n$, then f has a pole of order $n-(2m+4)$ and g has a zero of order $n+m-(2m+4)$, no further conditions.

In the last case we apply $T_U(t)$ and obtain a zero of f at z_0 of order $n-2-(2m+4)$. If this turns out to be zero, then $n = (2m+4)+2$ and g has a zero of order m at z_0. This gives a holomorphic potential.

Otherwise $n-2-(2m+4) > 0$ and for g we get the condition $m-n+2+(2m+4) < 0$. Thus we can apply again $T_V(t)$. From (3.8.1c) we see that we need to consider $k = 2m+2-n+2+(2m+4) = -n+2(2m+4)$. This leads again to three cases: If $k = 0$, then we have arrived at a holomorphic potential. If $k > 0$, then f has a zero of order k and g must satisfy $m-k < 0$. If $k < 0$, then f has a pole of order $-k$ and we can apply $T_U(t)$ and then $T_V(t)$ again. The new k is then of the form $-n+3(2m+4)$. Therefore, choosing the integer r maximal, so that $-n+r(2m+4) < 0$, we obtain the following sequence of zero

and pole orders for f and g, $0 \leq s \leq r-1$,

$$\begin{array}{ccccccccc} n & \overset{T_U(t)}{\longrightarrow} & n-2 & \dots \overset{T_V(t)}{\longrightarrow} & \kappa(s) & \overset{T_U(t)}{\longrightarrow} & \kappa(s)-2 & \overset{T_V(t)}{\longrightarrow} & \dots \\ n+m & \overset{T_U(t)}{\longrightarrow} & n-2-m & \dots \overset{T_V(t)}{\longrightarrow} & \kappa(s)+m & \overset{T_U(t)}{\longrightarrow} & \kappa(s)-2-m & \overset{T_V(t)}{\longrightarrow} & \dots \end{array}$$

Here, for notational convenience, we have set

$$\kappa(s) = n - s(2m+4). \tag{3.12.1}$$

For $s \leq r-1$, by the choice of r, all integers in the top row are positive. The integers in the bottom row are positive as well. For the numbers $n+m-s(2m+4)$ this follows from the positivity of the number above in the top row. For the numbers $n-2-m-s(2m+4)$ this is a consequence of the nonbranching assumption. We note that, by the choice of r, we can apply T_U at any rate.

There are two cases:

Case I: Transformation with $T_U(t)$ gives an f which has neither a pole nor a zero at z_0, i.e. $n-2-r(2m+4)=0$. In this case g has a zero of order m at z_0 and we get a locally holomorphic potential.

Case II: Transformation with $T_U(t)$ gives a zero of f of order $n-2-r(2m+4)>0$. Under this assumption we get a condition ensuring that g has a pole. This condition allows us to apply (3.8.1c). We see that the pole case does not occur because of the choice of r. Therefore we have two subcases to consider. We would also like to point out, that the condition on g, $n-m-2-r(2m+4)>0$, implies all the conditions on the bottom row above. Therefore, there will be no additional conditions to be considered.

Case IIa: $n-(r+1)(2m+4)=0$ or

Case IIb: $n-(r+1)(2m+4)<0$

In case IIa we get that g has a pole of order $m+2$ and f a zero of order $2m+2$. This allows us to apply (3.8.1c). We see that $T_V(t)$ yields a locally holomorphic f with $f(z_0) \neq 0$, whence a locally holomorphic potential.

In case IIb smoothness requires a pole of g, i.e. $n-m-2-r(2m+4)>0$. Because of $n-(r+1)(2m+4)<0$, (3.8.1c) shows, that $T_V(t)f$ has again a zero. This implies again a pole of the transformed g: $n+m-(r+1)(2m+4) = n-m-4-r(2m+4)<0$. These two conditions for the order of g can only be satisfied simultaneously if $n-m-2-r(2m+4)=1$ But this gives a simple pole of g in one of the dressing steps, which is impossible for a meromorphically integrable potential. This contradicts the assumption, that the original f and E described a CMC immersion. Therefore, this case cannot occur.

As a consequence, the only possible cases are

$$n = r(2m+4) \qquad \text{or} \qquad n = r(2m+4)+2, \tag{3.12.2}$$

for some integer number $r \geq 0$.

If we start with an f that has a zero of order $n>m$ at $z=z_0$, then the dressing transformation $T_V(t)$ gives a new $\hat{f}$. According to (3.8.1c) we have to consider $2m+2-n$. If $n<2m+2$, we get a zero of $\hat{f}$ and smoothness implies that $-n+m+2>0$. If $n=2m+2$

we get a locally holomorphic potential. The last case $n > 2m+2$ gives a pole of order $n-(2m+2) > 0$ of $\hat{f}$.

In the last case we can apply what we have shown above and see that $n-(2m+2) = r(2m+4)$ or $n-(2m+2) = r(2m+4)+2$ for some integer $r \geq 1$. Altogether this yields for n the two possibilities

$$n = r(2m+4) \qquad \text{or} \qquad n = r(2m+4) - 2 \tag{3.12.3}$$

for some integer number $r \geq 1$.
Therefore, only $n < 2m+2$ remains. In this case also $-n+m+2 > 0$, which is stronger than the first inequality. From the outset we have $n > m$. These two inequalities $n-m > 0$, $n-m-2 < 0$ imply $n-m = 1$. But this is the pole order of the original g, a contradiction, which implies that this case cannot occur.

3.13 Altogether we have shown

Theorem: *Let*

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & f(z) \\ \frac{E(z)}{f(z)} & 0 \end{pmatrix} \mathrm{d}z \in \mathcal{M}.$$

Then for ξ to yield a smooth CMC immersion under the DPW construction it is necessary and sufficient that for every $z_0 \in D$, where f has a pole or a zero, the following conditions are satisfied: Let m denote the zero order of E at z_0. If f has a pole of order n at z_0, then $n = 2$ or for some integer $r \geq 1$

$$n = r(2m+4) \qquad \textit{or} \qquad n = r(2m+4) + 2. \tag{3.13.1}$$

If f has a zero of order n at z_0, then for some integer $r \geq 1$

$$n = r(2m+4) \qquad \textit{or} \qquad n = r(2m+4) - 2. \tag{3.13.2}$$

Corollary 1: *If $\xi \in \mathcal{M}$ and E has no zeroes on D, then ξ yields a CMC immersion.*

Proof: Here $m = 0$, whence (3.13.1) and (3.13.2) only say, that f has only zeroes and poles of even order. But this is already part of $\xi \in \mathcal{M}$. □

Remark:

1. The conditions stated in the theorem above seem to be much more restrictive than the integrability conditions given in the last chapter.

2. The conditions in the Corollary 3.11 cannot be satisfied for a pair n, m, which satisfies the conditions of Theorem 3.13. This can easily be proved directly.

Finally we have the following interesting result:

Corollary 2: *Let ξ be a meromorphic potential of the form (1.1.1), where $E = f \cdot g$ is a holomorphic function and f is a meromorphic function without a pole at $z = 0$. If*

furthermore f has no zeroes and only poles of second order with vanishing residues, then ξ is associated with a CMC immersion.

Proof: By Corollary 2.6 and the assumptions on the coefficients of ξ, we get that $\xi \in \mathcal{M}$. Then the corollary follows immediately from the preceding theorem. □

4 Three examples

4.1 In this chapter we construct two examples in the dressing orbit of the cylinder,

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathrm{d}z, \tag{4.1.1}$$

and one example in a different dressing orbit.

Consider ξ as above. In this case, because $b_1(z) = z$ has no zeroes except at $z = 0$, we can use the transformation $T_U(t_0)$ to generate a pole of order 2 at the point $z = z_0 \neq 0$ if we choose $t_0 = -\frac{1}{b_1(z_0)}$.

The resulting meromorphic potential is of the form (1.1.1) with $E(z) = 1$ and

$$f(z) = \left(\frac{z_0}{z_0 - z}\right)^2, \ g(z) = f(z)^{-1}. \tag{4.1.2}$$

It belongs to a smooth surface without umbilics.

In Figure 1 we show part of the surface associated with $z_0 = \frac{1}{4}$.

Figure 1:

4.2 Next we would like to use $T_V(t)$ for some t to generate an additional zero at a point z_1. We need to compute

$$c_1(z) = \int_0^z \frac{E(z)}{f(z)} \mathrm{d}z = \frac{z^3 - 3z_0 z^2 + 3z_0^2 z}{3z_0^2}. \tag{4.2.1}$$

This function has three zeroes, one at $z = 0$ and the other two at $z = z_0(1 - e^{\pm\frac{2\pi}{3}})$. Except for z_1 being one of these points we can, by (3.8.2b), using $T_V(t)$ to add a zero at z_1 if we choose

$$t = t_0 = -c_1(z_1)^{-1}. \tag{4.2.2}$$

It turns out, that we get not just one, but rather three zeroes of $\hat{f} = T_V(t_0)f$, one at $z = z_1$, the other two at $z = z_0 + (z_1 - z_0)(1 - e^{\pm\frac{2\pi i}{3}})$. A straightforward calculation using (4.2.2) and (3.6.6) shows that

$$\hat{f}(z) = \left(\frac{z_0((z_0 - z_1)^3 - (z_0 - z)^3)}{(z_0 - z)((z_0 - z_1)^3 - z_0^3)}\right)^2.$$

4.3 As a last example we consider a meromorpic potential with a pole of sixth order at some $z = z_0 \neq 0$. If $f(z) = (z - z_0)^{-6}$, then we know from Theorem 3.13 that the Hopf differential must be either of order 0 or 1 at $z = z_0$. Let us take $E(z) = z - z_0$. This E automatically satisfies the meromorphic integrability condition (2.4.5) as was explained in the examples in section 2.12. Since in addition $n = 6$ and $m = 1$ satisfy the condition of Theorem 3.13, we get that this meromorphic potential is associated with a CMC immersion.

The surface associated with this meromorphic potential for $z_0 = \frac{1}{2}$ is partially shown in figure 2.

Figure 2:

A Appendix

In this appendix we will review the Sym-Bobenko formula and explain our conventions, which differ for example from those used in [1].

A.1 Let $\Phi : D \to \mathbb{R}^3$ be an immersion, $D \subset \mathbb{C}$ the open unit disk or the whole complex plane. If we take Φ as a conformal chart on the surface $\Phi(D)$, and the metric on $\Phi(D)$ as $\mathrm{d}s^2 = e^u(\mathrm{d}x^2 + \mathrm{d}y^2)$, $u : D \to \mathbb{R}$, then the natural frame,

$$(\Phi_x, \Phi_y, N), \qquad\qquad N = \frac{\Phi_x \times \Phi_y}{|\Phi_x \times \Phi_y|} \tag{A.1.1}$$

associated with Φ satisfies

$$\langle \Phi_x, \Phi_x \rangle = \langle \Phi_y, \Phi_y \rangle = e^u, \qquad\qquad \langle \Phi_x, \Phi_y \rangle = 0. \tag{A.1.2}$$

This shows that (Φ_x, Φ_y, N) is an orthogonal frame and

$$\mathcal{U} = (e^{-\frac{u}{2}}\Phi_x, e^{-\frac{u}{2}}\Phi_y, N) \tag{A.1.3}$$

is an orthogonal matrix. By possibly rotating the surface, we can assume $\mathcal{U}(0,0) = I$. We will use this normalization from now on.

We choose complex coordinates $z = x + iy$, $\bar{z} = x - iy$. Then $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ and

$$\langle \Phi_z, \Phi_z \rangle = \langle \Phi_{\bar{z}}, \Phi_{\bar{z}} \rangle = 0, \qquad\qquad \langle \Phi_z, \Phi_{\bar{z}} \rangle = \frac{1}{2}e^u. \tag{A.1.4}$$

Using the definitions

$$Q = \langle \Phi_{zz}, N \rangle, \qquad\qquad H = 2e^{-u}\langle \Phi_{z\bar{z}}, N \rangle \tag{A.1.5}$$

we can write the Weingarten map (conventions as in [9]) as

$$II \cdot I^{-1} = \begin{pmatrix} (Q + \bar{Q})e^{-u} + H & i(Q - \bar{Q})e^{-u} \\ i(Q - \bar{Q})e^{-u} & -(Q + \bar{Q})e^{-u} + H \end{pmatrix} \tag{A.1.6}$$

Therefore the mean curvature is given as $\frac{1}{2}\mathrm{tr}(II \cdot I^{-1}) = H$, justifying the nomenclature, and the Gauß curvature is given by $K = \det(II \cdot I^{-1}) = H^2 - 4(Q\bar{Q})e^{-2u}$.

A.2 In this section we discuss briefly the system of equations

$$\mathcal{U}^{-1}\mathcal{U}_z = A \qquad \text{and} \qquad \mathcal{U}^{-1}\mathcal{U}_{\bar{z}} = B. \tag{A.2.1}$$

This means we need to compute $\partial_z(e^{-\frac{u}{2}}\Phi_x)$, $\partial_z(e^{-\frac{u}{2}}\Phi_y)$ and $\partial_z N$, and the corresponding expressions for $\partial_{\bar{z}}$. A somewhat tedious and lengthy but standard computation yields $\partial_z\mathcal{U} = \mathcal{U}A$, where

$$A = \begin{pmatrix} 0 & \frac{i}{2}u_z & -(Q + \frac{1}{2}e^u H)e^{-\frac{u}{2}} \\ -\frac{i}{2}u_z & 0 & -i(Q - \frac{1}{2}e^u H)e^{-\frac{u}{2}} \\ (Q + \frac{1}{2}e^u H)e^{-\frac{u}{2}} & i(Q - \frac{1}{2}e^u H)e^{-\frac{u}{2}} & 0 \end{pmatrix}. \tag{A.2.2}$$

Then $\partial_{\bar{z}}\mathcal{U} = \mathcal{U}B$, where $B = \overline{A}$.

Theorem: *The system of equations $\partial_z\mathcal{U} = \mathcal{U}A$ and $\partial_{\bar{z}}\mathcal{U} = \mathcal{U}B$ has the compatibility condition*

$$A_{\bar{z}} - B_z - [A, B] = 0. \tag{A.2.3}$$

This is equivalent with

$$u_{z\bar{z}} + \frac{1}{2}e^u H^2 - 2e^{-u}|Q|^2 \quad = \quad 0, \tag{A.2.4}$$

$$Q_{\bar{z}} = \frac{1}{2}e^u H_z. \tag{A.2.5}$$

<u>Proof:</u> Straightforward.

A.3 In the following we adopt the spinor representation for vectors in $\mathbb{R}^3$, i.e. we identify $\mathbb{R}^3$ with $\mathbf{su}(2)$.

The map $J : \mathbb{R}^3 \to \mathbf{su}(2)$ is defined by $J\mathbf{r} \mapsto -\frac{i}{2}\mathbf{r}\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is the vector, whose components are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.3.1}$$

Therefore

$$J(x, y, z) = \frac{1}{2}\begin{pmatrix} -iz & -ix - y \\ -ix + y & iz \end{pmatrix}, \tag{A.3.2}$$

and we have

$$\langle \mathbf{r}_1, \mathbf{r}_2 \rangle = -2\mathrm{tr}(J\mathbf{r}_1 \cdot J\mathbf{r}_2) \tag{A.3.3}$$

and

$$J(\mathbf{r}_1 \times \mathbf{r}_2) = [J\mathbf{r}_1, J\mathbf{r}_2]. \tag{A.3.4}$$

This means, that J is an isomorphism of $\mathbb{R}^3$ equipped with the crossproduct and $\mathbf{su}(2)$. The formula above states, that the natural scalar product of $\mathbb{R}^3$ corresponds via J, up to a factor, to the (invariant) Killing form of the semisimple Lie algebra $\mathbf{su}(2)$.

Clearly, for every linear map A of $\mathbb{R}^3$ the map JAJ^{-1} is a linear map of $\mathbf{su}(2)$, and every linear map of $\mathbf{su}(2)$ is of this form.

For every invertible linear map of $\mathbb{R}^3$ we know

$$(A\mathbf{r}_1) \times (A\mathbf{r}_2) = (\det A)(A^{-1})^\top(\mathbf{r}_1 \times \mathbf{r}_2). \tag{A.3.5}$$

Therefore, the group of automorphisms of $\mathbb{R}^3$ equipped with the cross product is the group

$$G = \{A \in \mathbf{GL}(3,\mathbb{R});\ A = \det A(A^{-1})^\top\}. \tag{A.3.6}$$

The defining equation implies $\det A = (\det A)^3(\det A)^{-1}$, whence $\det A = 1$ and $A = (A^{-1})^\top$. Therefore,

$$G = \mathbf{SO}(3). \tag{A.3.7}$$

This proves that the group $\mathrm{Aut}(\mathbf{su}(2))$ of automorphisms of $\mathbf{su}(2)$ is given by

$$\mathrm{Aut}(\mathbf{su}(2)) = J\mathbf{SO}(3)J^{-1}. \tag{A.3.8}$$

In particular, $\mathrm{Aut}(\mathbf{su}(2))$ is connected. On the other hand, there is a natural map $\mathbf{SU}(2) \to \mathrm{Aut}(\mathbf{su}(2))$, where $P \in \mathbf{SU}(2)$ acts as an inner automorphism $X \mapsto PXP^{-1}$. Since also $\mathbf{SU}(2)$ is connected, the latter map is surjective.

Theorem: *There exists a smooth map $P : D \to \mathbf{SU}(2)$, such that the following diagram commutes*

$$\begin{tikzcd}
& \mathbf{SU}(2) \arrow[dr] & \\
D \arrow[ur, "P"] \arrow[r, "\mathcal{U}"] & \mathbf{SO}(3) \arrow[r] & \mathrm{Aut}(\mathbf{su}(2))
\end{tikzcd}$$

where the unlabeled maps have been defined above. Moreover, P is unique up to a sign.

Proof: Denote by $\hat{\mathcal{U}}$ the composition of the two maps in the bottom row. Since D and $\mathbf{SU}(2)$ are simply connected, this map factors through $\mathbf{SU}(2)$. $\square$

Corollary: *For every $X \in \mathbf{su}(2)$ there holds*

$$(J\mathcal{U}J^{-1})(X) = PXP^{-1}.$$

A.4 We extend J to a $\mathbb{C}$-linear map from $\mathbb{C}^3$ to $\mathbb{C} \otimes \mathbf{su}(2) \cong \mathbf{sl}(2,\mathbb{C})$. Then the complex conjugation $\mathbf{r} \mapsto \overline{\mathbf{r}}$ of $\mathbb{C}^3$ satisfies

$$J\overline{\mathbf{r}} = -\overline{(J\mathbf{r})}^\top. \tag{A.4.1}$$

Using this and Theorem A.2 we show

Proposition:

(1) $(J\mathcal{U}_z J^{-1})(X) = [P_z P^{-1}, PXP^{-1}]$,

(2) $(J\mathcal{U}_{\bar{z}} J^{-1})(X) = [P_{\bar{z}} P^{-1}, PXP^{-1}]$,

(3) $(J(\mathcal{U}^{-1}\mathcal{U}_z)J^{-1})(X) = [P^{-1}P_z, X]$,

(4) $(J(\mathcal{U}^{-1}\mathcal{U}_{\bar{z}})J^{-1})(X) = [P^{-1}P_{\bar{z}}, X]$.

Proof: These relations follow by a straightforward computation from Corollary A.3. □

The proposition above suggests to translate from a linear 3×3-system for $\mathcal{U}$ to a linear 2×2-system for P.

Theorem: *For the map $P : D \to \mathbf{SU}(2)$ we have*

(1)

$$U = P^{-1}P_z = \begin{pmatrix} -\frac{1}{4}u_z & Qe^{-\frac{u}{2}} \\ -\frac{1}{2}e^{\frac{u}{2}}H & \frac{1}{4}u_z \end{pmatrix},$$

(2)

$$V = P^{-1}P_{\bar{z}} = \begin{pmatrix} \frac{1}{4}u_{\bar{z}} & \frac{1}{2}He^{\frac{u}{2}} \\ -\bar{Q}e^{-\frac{u}{2}} & -\frac{1}{4}u_{\bar{z}} \end{pmatrix},$$

(3) $P(0,0) = I$.

Moreover, the compatibility condition for (1) and (2) is the same as the one for (A.2.1).

Proof: In section A.2 we have stated the form of $\mathcal{U}^{-1}\mathcal{U}_z$ and $\mathcal{U}^{-1}\mathcal{U}_{\bar{z}}$. This makes it straightforward to compute $(J(\mathcal{U}^{-1}\mathcal{U}_z)J^{-1})(X)$ and $(J(\mathcal{U}^{-1}\mathcal{U}_{\bar{z}})J^{-1})(X)$. Comparing the resulting expressions with $[R, X]$ produces (1) and (2). The equation in (3) follows from our normalization $\mathcal{U}(0,0) = I$. The last claim is clear, since J is an isomorphism. □

A.5 Similar to Corollary A.3 we state

$$J(\mathcal{U}e_j) = (J\mathcal{U}J^{-1})(Je_j). \tag{A.5.1}$$

We note that the definition of J and σ_j implies

$$Je_j = -\frac{i}{2}\sigma_j. \tag{A.5.2}$$

This shows

$$J(e^{-\frac{u}{2}}\Phi_x) = -\frac{i}{2}P\sigma_1 P^{-1}, \tag{A.5.3}$$

$$J(e^{-\frac{u}{2}}\Phi_y) = -\frac{i}{2}P\sigma_2 P^{-1}, \tag{A.5.4}$$

$$J(N) = -\frac{i}{2}P\sigma_3 P^{-1}. \tag{A.5.5}$$

As a consequence we obtain

$$J\Phi_z = -\frac{i}{2}e^{\frac{u}{2}}P\sigma_-P^{-1}, \tag{A.5.6}$$

$$J\Phi_{\bar z} = -\frac{i}{2}e^{\frac{u}{2}}P\sigma_+P^{-1}, \tag{A.5.7}$$

where

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{A.5.8}$$

A.6 We look at the compatibility conditions stated in Theorem A.2 a bit more closely. We note that if H is constant, then Q is holomorphic. $Q\mathrm{d}z^2$ is therefore a holomorphic differential for CMC immersions, the Hopf differential. Moreover, for differentiable surfaces condition (A.2.4) is automatically satisfied.

By the standard theory of Bianchi for CMC surfaces, we know, that to every CMC immersion Φ, there belongs an associated family of CMC surfaces. These are classically defined by the substitution $Q \mapsto \lambda^{-2}Q$, where $\lambda = e^{it}$ is on the unit circle in $\mathbb{C}$.

If a function u solves (A.2.4) for Q, then the same u solves it for $\lambda^{-2}Q$. Therefore, to the same solution u there belongs a one parameter family of surfaces, which are determined by the integral $P(\lambda)$, which depends on λ through Q.

We would also like to point out that in the DPW method we work with matrices in a twisted loop group. The introduction of λ in the classical way doesn't give elements in this group. We actually have

$$U(\lambda) = \begin{pmatrix} -\frac{1}{4}u_z & \lambda^{-2}Qe^{-\frac{u}{2}} \\ -\frac{1}{2}He^{\frac{u}{2}} & \frac{1}{4}u_z \end{pmatrix}, \tag{A.6.1}$$

$$V(\lambda) = \begin{pmatrix} \frac{1}{4}u_{\bar z} & \frac{1}{2}He^{\frac{u}{2}} \\ -\lambda^2\bar{Q}e^{-\frac{u}{2}} & -\frac{1}{4}u_{\bar z} \end{pmatrix}. \tag{A.6.2}$$

By conjugation with the z-independent matrix

$$G = i\begin{pmatrix} 0 & \lambda^{-\frac{1}{2}} \\ \lambda^{\frac{1}{2}} & 0 \end{pmatrix} \tag{A.6.3}$$

we get elements $\hat{U}(\lambda)$ and $\hat{V}(\lambda)$ in the twisted loop algebra $\Lambda\mathbf{sl}(2,\mathbb{C})$: $P \mapsto F = G^{-1}PG$, and

$$U \mapsto \hat{U} = G^{-1}UG = \begin{pmatrix} \frac{1}{4}u_z & -\frac{1}{2}\lambda^{-1}He^{\frac{u}{2}} \\ \lambda^{-1}Qe^{-\frac{u}{2}} & -\frac{1}{4}u_z \end{pmatrix}, \tag{A.6.4}$$

$$V \mapsto \hat{V} = G^{-1}VG = \begin{pmatrix} -\frac{1}{4}u_{\bar z} & -\lambda\bar{Q}e^{-\frac{u}{2}} \\ \frac{1}{2}\lambda He^{\frac{u}{2}} & \frac{1}{4}u_{\bar z} \end{pmatrix}. \tag{A.6.5}$$

We note that $F(0,0,\lambda) = I$, therefore F lies in the twisted loop group $\Lambda\mathbf{SU}(2)$

A.7 By the transformations carried out so far, the frame $\mathcal{U}$ has been translated into the "extended lift" F. In this setting the Sym-Bobenko formula provides an easy way to derive the immersion Φ_λ from the extended lift $F(\lambda)$:

Theorem: *Let $\Phi_\lambda : D \to \mathbb{R}^3$, D as above, be an associated family of CMC immersions with mean curvature H. Let $G(\lambda)$ be defined by (A.6.3) and $F(\lambda)$ be given by the definitions in section A.6, then*

$$G(\lambda)^{-1}J(\Phi_\lambda)G(\lambda) = -\frac{1}{2H}\left(\frac{\partial F}{\partial t}F^{-1} + \frac{i}{2}F\sigma_3F^{-1}\right) + C,\ \lambda = e^{it},$$

where, for each $\lambda \in S^1$, $C = C(\lambda)$ is a z-independent translation of the whole surface.

Proof: Let "$\cdot$" denote differentiation w.r.t. t. Then

$$\begin{aligned}
&\partial_z\left(-\frac{1}{2H}\left(\dot{F}F^{-1} + \frac{i}{2}F\sigma_3F^{-1}\right)\right) = \\
&= -\frac{1}{2H}(\dot{F}_zF^{-1} - \dot{F}F^{-1}F_zF^{-1} + \frac{i}{2}F_z\sigma_3F^{-1} - \frac{i}{2}F\sigma_3F^{-1}F_zF^{-1}) \\
&= -\frac{1}{2H}\mathrm{Ad}F(\dot{\hat{U}} + \frac{i}{2}[\hat{U}, \sigma_3]) \\
&= -\frac{1}{2}\mathrm{Ad}F\left(\begin{pmatrix} 0 & \frac{i}{2}\lambda^{-1}e^{\frac{u}{2}} \\ -i\frac{Q}{H}\lambda^{-1}e^{-\frac{u}{2}} & 0 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} 0 & \lambda^{-1}e^{\frac{u}{2}} \\ \frac{2}{H}\lambda^{-1}Qe^{-\frac{u}{2}} & 0 \end{pmatrix}\right) \\
&= -\frac{i}{2}e^{\frac{u}{2}}\lambda^{-1}\mathrm{Ad}F\sigma_+.
\end{aligned}$$

On the other hand from section A.5 we know

$$J((\Phi_\lambda)_z) = -\frac{i}{2}e^{\frac{u}{2}}P\sigma_-P^{-1} = -\frac{i}{2}e^{\frac{u}{2}}GFG^{-1}\sigma_-GF^{-1}G^{-1} = -\frac{i}{2}e^{\frac{u}{2}}\lambda^{-1}G(\mathrm{Ad}F\sigma_+)G^{-1}. \tag{A.7.1}$$

Similarly we get

$$G(\lambda)^{-1}J((\Phi_\lambda)_{\bar{z}})G(\lambda) = -\frac{i}{2}e^{\frac{u}{2}}\lambda\mathrm{Ad}F\sigma_- = \partial_{\bar{z}}\left(-\frac{1}{2H}\left(\dot{F}F^{-1} + \frac{i}{2}F\sigma_3F^{-1}\right)\right). \tag{A.7.2}$$

This proves that the two sides of the claim can only differ by a constant, which amounts to a translation of the whole surface. □

Remark: We actually have, that if we define $\widetilde{\Phi}_\lambda : D \to \mathbb{R}^3$ by

$$J(\widetilde{\Phi}_\lambda) = -\frac{1}{2H}\left(\frac{\partial F}{\partial t}F^{-1} + \frac{i}{2}F\sigma_3F^{-1}\right), \tag{A.7.3}$$

then $\widetilde{\Phi}_\lambda(z=0) = \frac{1}{2H}e_3$. The maps $\widetilde{\Phi}_\lambda$ also define an associated family of CMC immersions, which is related to the family Φ_λ by a λ-dependent Euclidean transformation in $\mathbb{R}^3$.

A.8 In the DPW approach F is obtained by an Iwasawa decomposition of g_-. In this case, by [5, Lemma 4.2], we get, that the derivative of the Gauß map $F^{-1}\mathrm{d}F$ has the form

$$F^{-1}\mathrm{d}F = \lambda^{-1}\alpha_1'\mathrm{d}z + \alpha_0 + \lambda\alpha_1''\mathrm{d}\bar{z}, \tag{A.8.1}$$

where α_0 is a one form with values in the diagonal elements in $\mathbf{sl}(2,\mathbb{C})$ and $\alpha_1'\mathrm{d}z$, $\alpha_1''\mathrm{d}\bar{z}$ are the holomorphic and antiholomorphic part, respectively, of a one form taking values in the off-diagonal matrices in $\mathbf{sl}(2,\mathbb{C})$.

Now we prove that Φ fails to be an immersion, where the meromorphic potential ξ is holomorphic and the upper right corner $f(z)$ of ξ has a zero.

Theorem: *If a CMC immersion is obtained by the DPW procedure starting from a meromorphic potential ξ of the form (1.1.1), then the upper right entry of ξ cannot have a zero at a point, where ξ is defined.*

Proof: Let ξ be a meromorphic potential and assume, that it is associated with a smooth CMC map. Then the canonical moving frame $\hat{F} \in \Lambda\mathbf{SU}(2)_\sigma$ obtained via Iwasawa decomposition satisfies $g_- = \hat{F}\hat{g}_+^{-1}$, $\hat{g}_-^{-1}\mathrm{d}\hat{g}_- = \xi$, $\hat{g}_-(0,0,\lambda) = I$ and $\hat{g}_+ \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$. On the other hand, the decompositions used in the body of this paper are obtained as follows: one integrates $g_-' = g_-\xi$, $g_-(0,0,\lambda) = I$ and decomposes $g_- = Fg_+^{-1}$, where $F \in \Lambda\mathbf{SU}(2)_\sigma$ and $g_+ \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$.

It is easy to see that $\hat{F} = Fk$ holds, where $k = k(z,\bar{z})$ is a λ-independent diagonal matrix. If ξ is defined at z_0, then all occuring matrices are smooth in a neighbourhood of z_0. We can therefore assume $\hat{F} = F$. This allows us to use the (proof of the) Sym-Bobenko formula.

We will actually prove, that the derivative $\mathrm{d}\Phi(z_0)$ vanishes if $f(z_0) = 0$. From the proof of the Sym-Bobenko formula above we know

$$J((\Phi_\lambda)_z) = -\frac{i}{2}e^{\frac{u}{2}}\lambda^{-1}\mathrm{Ad}F\sigma_+. \tag{A.8.2}$$

Here $s = -\frac{1}{2}\lambda^{-1}He^{\frac{u}{2}}$ is the upper right entry of α_1', which can be read off equation (A.6.4). On the other hand, because of $F = g_-g_+$, we get that the upper right entry of $F^{-1}F_z$ is of the form

$$\lambda^{-1}w(z,\bar{z})^{-2}f(z) + \lambda^1\ldots, \tag{A.8.3}$$

where $w(z,\bar{z})$ is the coefficient of λ^0 in the upper diagonal matrix entry of g_+. If ξ is defined at z_0, then g_-, g_+ and F are smooth at z_0. Therefore, with $w(z,\bar{z}) \neq 0$, $(\Phi_\lambda)_z$ vanishes at z_0 if and only if f does. Similarly we get that $(\Phi_\lambda)_{\bar{z}}$ vanishes at z_0 if and only if f does, which proves the theorem. $\square$

Remark: The proof above shows that

$$w^{-2}f = -\frac{1}{2}He^{\frac{u}{2}}. \tag{A.8.4}$$

The lower left entry of $F^{-1}F_z$ is

$$\lambda^{-1}w^2\frac{E}{f} + \lambda^1\ldots \tag{A.8.5}$$

which gives, by equation (A.6.4),

$$w^2\frac{E}{f} = Qe^{-\frac{u}{2}}. \tag{A.8.6}$$

Using this and (A.8.4) we get $E = -\frac{1}{2}QH$. By comparing this with the definition of the Hopf differential in [5, (4.39)], let us call it $\tilde{Q}$, and using [5, (4.41)] we see that with our definitions $\tilde{Q} = \langle N_z, N_z\rangle = -4E = 2HQ$. For $H = -2$ we have $Q = E$. Of course, sign and absolute value of H can be adjusted by the choice of the normal vector and/or a scaling of the metric in $\mathbb{R}^3$.

As another consequence we obtain a result emerging in a discussion of Burstall and Pedit with the authors.

Corollary: *Let ξ be a meromorphic potential of a CMC surface, and g_- as above. Let $g_- = \hat{F}\hat{g}_+^{-1}$ denote the Iwasawa decomposition of g_-, and $g_- = Fg_+^{-1}$ any decomposition of g_- with $g_+ \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$ and $F \in \Lambda\mathbf{SU}(2)_\sigma$ smooth. Furthermore, let $((g_+)_0)_{11} = ae^{ib}$ be the polar decomposition of the λ^0-coefficient of $\hat{g}_+$. Then $e^{-2ib}f$ is real.*

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